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Fractional branes and boundary states in orbifold theories

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Abstract: We study the D-brane spectrum of $\mathcal{N} = 2$ string orbifold theories using the boundary state formalism. The construction is carried out for orbifolds with isolated singularities, non-isolated singularities and orbifolds with discrete torsion. Our results agree with the corresponding K-theoretic predictions when they are available and generalize them when they are not. This suggests that the classification of boundary states provides a sort of "quantum K-theory" just as chiral rings in CFT provide "quantum" generalizations of cohomology. We discuss the identification of these states with D-branes wrapping holomorphic cycles in the large radius limit of the CFT moduli space. The example $\mathbb{C}^3/\mathbb{Z}_3$ is worked out in full detail using local mirror symmetry techniques. We find a precise correspondence between fractional branes at the orbifold point and configurations of D-branes described by vector bundles on the exceptional $\mathbb{P}^2$ cycle.

Keywords: D-branes, Conformal Field Models in String Theory.

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Contents

1. Introduction

The realization that D-branes [1] carry Ramond-Ramond charge [2] has been a crucial ingredient in recent progress in non-perturbative string theory. The simple CFT description of these states as hyperplanes on which open strings can end has provided new insights in the structure of space-time at substringy length scales [3] and has given an adequate framework for attempting a microscopic formulation [4] of M-theory. However, there are many vacua of string theory which do not admit a clear space-time interpretation — they are described by an abstract CFT — and for which the description of D-branes is not manifest. Moreover, these vacua are expected to contain non-perturbative states carrying Ramond-Ramond charge and to fit in the web of dualities relating the different string theories. It is therefore important to describe D-branes in general string vacua and at generic points of moduli space, where a space-time interpretation might not be obvious, in order to unravel the complete description of the physics of M-theory.

A powerful tool for analyzing the inclusion of boundaries on string worldsheets is the boundary state formalism [5]. This formalism provides a closed string description of D-branes and it is applicable, in principle, to arbitrary CFTs. In this vein, one can tackle the question of determining the D-brane spectrum at an arbitrary point of moduli space of string theory. Along different lines, Witten [6] has argued that the classification of D-branes in a general space-time background has to be upgraded from singular homology to K-theory. This generalization has brought to light a new understanding of D-branes and the appearance of new hitherto unsuspected states. We strongly believe that the classification of boundary states of a given CFT provides a stringy generalization of K-theory and that its study should provide a sort of "quantum K-theory" just as CFT chiral rings generalized cohomology to "quantum cohomology" [7]. Some hints of this will appear in this work. Moreover, the understanding of D-branes at a generic point in moduli space and D-geometry should shed new light on what space-time at the shortest length scales really is.

In this paper we classify D-brane states at certain points of the moduli space of type-II strings with eight supercharges. In particular we provide a boundary state description of D-brane states at points in moduli space that admit a perturbative orbifold CFT description.[1] This gives a closed string description of the fractional D-branes in [9]–[12]. As a byproduct of the construction of consistent boundary states, we give a physical D-brane proof of the McKay correspondence [13]–[20]. Roughly speaking, this correspondence states that there is a one-to-one correspondence between the number of non-trivial irreducible representations of the orbifold group Γ describing the isolated orbifold singularity $\mathbb{C}^d/\Gamma$ and the homology generators of the crepant resolution of the singularity. In section 5 this will be explained in the more precise language of K-theory. In our physical situation, this follows from a deformation of the BPS spectrum determined by boundary states to the large radius limit of the orbifold moduli space where the states can be realized as D-branes wrapping supersymmetric cycles. We also analyze boundary states for certain non-isolated singularities and find that D-brane considerations suggest a generalization of the McKay correspondence to this case. This would relate non-trivial irreducible representations of the discrete group to compact homology generators of the resolved space.

The reduced amount of supersymmetry of these vacua introduces corrections to the special geometry of the moduli space, which makes the abovementioned identification very difficult. In section 5 we work out an example by providing the exact solution for the $\mathbb{C}^3/\mathbb{Z}_3$ model using the techniques of local mirror symmetry [21]–

[1] Some previous work on boundary states and orbifolds can be found in [8].

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[24]. We identify the singularities in moduli space, the monodromies around these points and the exact central charge for the model. This allows us to make a precise identification between boundary states at the orbifold point with branes in the large volume limit. Moreover, the fractional branes at the orbifold are extended to an arbitrary point in moduli space. Similar results for the quintic moduli space have been obtained in [25]. The role of the perturbative orbifold point is played there by the Gepner point.

The boundary state formalism serves very useful in analyzing the spectrum of models that do not have an obvious geometrical interpretation such as orbifolds with discrete torsion [26, 27]. We construct the consistent set of boundary states corresponding to fractional branes for models with discrete torsion and find that these are classified by irreducible projective representations of the discrete group. Since orbifolds in this class do not admit complete smooth resolutions, there is no obvious geometric interpretation similar to McKay correspondence. The ease by which the boundary states generalize to these models indeed suggest that the most general framework for D-branes is the boundary state formalism. In certain situations, this conformal field theory construction reduces to a known mathematical structure such as cohomology or K-theory. We believe that this is also the case for orbifolds with discrete torsion, where the fractional branes should be described by an appropriate generalization of equivariant K-theory [6].

The paper is organized as follows. In section 2 we summarize the description of fractional branes at orbifold singularities and some of their properties from a D-brane probe perspective. In section 3 we give a selfcontained discussion of boundary states and construct the boundary states in flat space and set up the notation. In section 4 we give the general prescription required to write down boundary states for perturbative orbifold CFTs and give a general argument which predicts the number of states. We then go on and explicitly work out examples involving orbifolds with isolated singularities, with non-isolated singularities and with discrete torsion. In section 5 we elucidate the role of these boundary states and their relation to D-brane states at the large volume limit, and explain how the boundary state construction sheds new light onto the McKay correspondence. In section 6 we consider the $\mathbb{C}^3/\mathbb{Z}_3$ example and work out the exact solution of the model including worldsheet instanton corrections. We then make a precise mapping between fractional branes and D-branes in the large volume limit. We also extend these states to generic points in moduli space. In appendix we summarize conventions and some useful formulas.

2. Branes at orbifold singularities and fractional branes

The gauge theory on a D-brane probe reproduces the space-time in which it is embedded as its moduli space of vacua. It also captures many features of the space-time BPS states which correspond to wave functions on the moduli space of the brane.

When the brane theory is known, this can be an efficient tool in addressing dynamical problems. Hence, in these situations, it is very useful to establish an explicit correspondence between field theory and space-time excitations. A very rich background for testing these ideas is given by orbifold singularities with a perturbative conformal field theory description. In this section we will give a brief presentation of D-particle states in orbifold theories of the form $\mathbb{C}^d/\Gamma$, $\Gamma \subset \mathrm{SU}(d)$, from the probe perspective [28, 29]. The following sections will contain a more detailed description of the same states using the boundary state formalism.

The theory on D-branes probing a $\mathbb{C}^d/\Gamma$ singularity is uniquely determined by choosing a representation of[2] Γ, which defines the action on the Chan-Paton indices, and by specifying an action of Γ on $\mathbb{C}^d$ via a d-dimensional representation R. The bosonic projection equations that need to be solved are

$$\begin{aligned} \gamma(g_i) A \gamma(g_i)^{-1} &= A\,, \qquad i = 1, \ldots, |\Gamma|\,, \\ \gamma(g_i) Z^\alpha \gamma(g_i)^{-1} &= R(g_i)^\alpha_\beta Z^\beta\,, \qquad \alpha = 1, \ldots, d\,, \end{aligned} \tag{2.1}$$

where Z^α are the complex fields which parameterize $\mathbb{C}^d$. γ is any representation of the orbifold group Γ

$$\gamma = \oplus_{a=1}^n N_a \gamma_a\,, \tag{2.2}$$

consisting of N_a copies of the a-th irreducible representation γ_a of Γ. The gauge group of the D-brane theory is the commutant of γ in $\mathrm{U}(N)$, where $N = \sum_{a=1}^n d_a N_a$ with $d_a = \dim(\gamma_a)$, which yields a $G = \prod_{a=1}^n \mathrm{U}(N_a)$ gauge theory. The matter content can be similarly found and it is encoded in the representation theory of Γ. The gauge theory describes the physics of branes at the singularity.

Although this formalism is very general, in the following we specialize to D0-brane quantum mechanics. In the Born Oppenheimer approximation, the moduli space of vacua has two branches with different space-time interpretation. The Higgs branch — spanned by the expectation values of the fields Z^α — reproduces the orbifold geometry. From a dynamical point of view, excitations propagating along this branch correspond to D0-branes that can move about the singularity. The theory also has a Coulomb branch parameterized by expectation values of the fields ϕ^i in the vector multiplet, the Z^α being set to zero. The excitations along the Coulomb branch have been interpreted in [9]–[12] as branes wrapping the shrunken cycles of the orbifold singularity. There is a one-to-one correspondence between such BPS states and irreducible representations of the orbifold group Γ. If we resolve the orbifold singularity by turning on marginal operators, the same states can be identified with branes wrapping supersymmetric cycles in a smooth space-time background. This can be thought of as a physical McKay correspondence and it will be considered in more detail later.

[2]We will consider supersymmetric backgrounds so that Γ is a discrete subgroup of $\mathrm{SU}(d)$.

These states carry charge under the untwisted, as well as twisted RR fields. The values of the charges can be determined by an open string disk computation as outlined in [28, 11]. The state corresponding to the a-th irreducible representation has charge

$$Q_0^a = \frac{d_a}{|\Gamma|}, \qquad a = 1, \ldots, n \tag{2.3}$$

with respect to the untwisted RR field and

$$Q_m^a = \frac{\chi^a(g_m)}{|\Gamma|}, \qquad a = 1, \ldots, n \tag{2.4}$$

with respect to the m-th twisted RR field, where $\chi^a(g_m) = \mathrm{Tr}(\gamma_a(g_m))$ is the character of the a-th representation. Note that the D0-brane charge (2.3) is fractional.

A similar analysis can be made for orbifolds with discrete torsion. As shown in [30, 31], discrete torsion can be incorporated in the probe gauge theory by considering Chan-Paton factors in a projective representation of the gauge group. The cocycle factors appearing in the orbifold conformal field theory partition function [26, 27] are representatives of a cohomology class in $H^2(\Gamma, \mathrm{U}(1))$.

It can be shown using the methods of [32], that all discrete subgroups of SU(2) have trivial U(1)-valued cohomology, so discrete torsion cannot be implemented in $\mathbb{C}^2/\Gamma$ orbifolds.[3] Nontrivial examples can be found for $\Gamma \subset \mathrm{SU}(3)$ acting on $\mathbb{C}^3$. Given $\Gamma \in \mathrm{SU}(3)$ with non-trivial $H^2(\Gamma, \mathrm{U}(1))$, and a set of cocycle factors defining the discrete torsion, the gauge theory on the D-brane probe is found by solving (2.1) with a projective action of the orbifold group on Chan-Paton factors. The choice of a cohomology class in $H^2(\Gamma, \mathrm{U}(1))$ — specifying the discrete torsion phases — determines a class of irreducible projective representations of Γ modulo projective equivalence. Within this class, there are in general $m \neq 1$ (linear equivalence classes of) irreducible projective representations. These determine the gauge group to be $G = \prod_{b=1}^{m} U(N_b)$ just as before. An important difference here is that the number of projective representations, m, is not linked anymore with the number of conjugacy classes of Γ. The gauge theory probe has again m BPS states on the Coulomb branch. However, since the resolution of these orbifolds is not complete in string theory [27, 30, 31], these states cannot be given a clear geometric interpretation. We believe that they can be given nevertheless an appropriate boundary state and K-theoretic interpretation, but we will return to this later on.

In the next sections we will construct all these states using the boundary state formalism and reproduce their known properties extracted from the probe theory approach.

[3] This can also be easily seen by showing that all discrete subgroups of SU(2) do not admit non-trivial projective representations and taking into account the relation between $H^2(\Gamma, \mathrm{U}(1))$ and classes of projective representations mentioned below.

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3. D-branes as boundary states

In the next section we will construct boundary states corresponding to the fractional branes introduced in the previous discussion. For completeness we introduce here the construction of boundary states in flat space and set up the notation.

3.1 General construction

Boundary states describe in the closed string theory language the inclusion of boundaries on the worldsheet [5]. On these boundaries, conformally invariant boundary conditions are imposed on the worldsheet closed string fields. The eigenstates of these boundary conditions are called Ishibashi states. Consistency of factorization with the open string channel [33] select appropriate linear combinations of Ishibashi states, which are called boundary states. Boundary states provide a complementary description of D-branes, which have a simple open string description when a space-time interpretation is available. It is therefore important to analyze boundary states for generic CFTs since many vacua of string theory do not admit a clear space-time interpretation.

The non-linear constraints imposed by conformal invariance on Ishibashi states are usually solved by replacing them by linear constraints. The cylinder amplitude describing propagation of a boundary state into another must admit an open string interpretation as a one-loop vacuum amplitude with integer degeneracy of states at any given mass level. This constraint, usually referred as Cardy's condition [33], severely restricts the allowed linear combinations of Ishibashi states.

For a general CFT with a boundary, conservation of momentum across the boundary imposes the following condition

$$T_L(t,\sigma)|_{t=0} = T_R(t,\sigma)|_{t=0}\,, \tag{3.1}$$

which relates the left-moving and right-moving stress energy tensors of the "bulk" CFT at the $t=0$ boundary. For a CFT with a more general left-right symmetric chiral algebra $\mathcal{A}$ than the Virasoro algebra, one may identify the extra symmetry generators on the left with those on the right at the boundary up to the action of an automorphism of $\mathcal{A}$.[4] For the CFT describing string theory in flat space-time, vanishing of momentum along the boundary requires[5]

$$T_{t\sigma}|_{t=0} = \partial_t X^M \partial_\sigma X_M + i\widetilde{\psi}^M \partial_+ \widetilde{\psi}_M - i\psi^M \partial_- \psi_M|_{t=0} = 0\,, \qquad M=0,\ldots,9 \tag{3.2}$$

which is satisfied by imposing the linear conditions

$$\begin{aligned} \partial_t X|_{t=0} = 0 &\Longrightarrow (\partial_- X + \partial_+ X)|_{t=0} = 0 \qquad \text{Neumann}\,, \\ \partial_\sigma X|_{t=0} = 0 &\Longrightarrow (\partial_- X - \partial_+ X)|_{t=0} = 0 \qquad \text{Dirichlet} \end{aligned} \tag{3.3}$$

[4] For a nice discussion on this see [34].
[5] $\partial_\pm$ refers to derivatives with respect to world-sheet light-cone coordinates $t^\pm = t \pm \sigma$.

corresponding to Neumann and Dirichlet boundary conditions, respectively. Superconformal invariance on the worldsheet also relates the left moving supersymmetry generator with the right moving one

$$\left(T_L^F - i\eta T_R^F\right)\big|_{t=0} = 0\,, \tag{3.4}$$

where $\eta = \pm 1$ labels the spin structure. Since $T_L^F = \psi^M \partial_- X_M$, $T_R = \widetilde{\psi}^M \partial_+ X_M$ and (3.3) the Neumann and Dirichlet boundary conditions on worldsheet fermions are

$$(\psi + i\eta\widetilde{\psi})|_{t=0} = 0 \qquad \text{Neumann}\,, \qquad (\psi - i\eta\widetilde{\psi})|_{t=0} = 0 \qquad \text{Dirichlet} \tag{3.5}$$

which also solve (3.2).

We will now write down the boundary states corresponding to D-branes in the type-II theories and introduce the necessary ingredients to build boundary states for orbifolds.[6] For simplicity, we will fix the light-cone gauge by taking as $x^8 \pm x^9$ as light-cone coordinates [37] after a double Wick rotation on x^0 and x^8. The boundary conditions on the closed string world-sheet fields for a Dp-brane are[7]

$$\begin{aligned}
\partial_t X^\mu(t=0,\sigma)|\eta,k\rangle_{\substack{\text{NSNS}\\ \text{RR}}} &= 0 \Longrightarrow (\alpha_n^\mu + \widetilde{\alpha_{-n}}^\mu)|\eta,k\rangle_{\substack{\text{NSNS}\\ \text{RR}}} = 0\,, \qquad \mu = 0,\ldots,p\,,\\
\partial_\sigma X^i(t=0,\sigma)|\eta,k\rangle_{\substack{\text{NSNS}\\ \text{RR}}} &= 0 \Longrightarrow (\alpha_n^i - \widetilde{\alpha_{-n}}^i)|\eta,k\rangle_{\substack{\text{NSNS}\\ \text{RR}}} = 0\,, \qquad i = p+1,\ldots,7\,,\\
(\psi^\mu + i\eta\widetilde{\psi}^\mu)|\eta,k\rangle_{\substack{\text{NSNS}\\ \text{RR}}} &= 0 \Longrightarrow (\psi_r^\mu + i\eta\widetilde{\psi_{-r}}^\mu)|\eta,k\rangle_{\substack{\text{NSNS}\\ \text{RR}}} = 0\,,\\
(\psi^i - i\eta\widetilde{\psi}^i)|\eta,k\rangle_{\substack{\text{NSNS}\\ \text{RR}}} &= 0 \Longrightarrow (\psi_r^i - i\eta\widetilde{\psi_{-r}}^i)|\eta,k\rangle_{\substack{\text{NSNS}\\ \text{RR}}} = 0\,,\\
x^{1,2}|\eta,k\rangle_{\substack{\text{NSNS}\\ \text{RR}}} &= 0\,,
\end{aligned} \tag{3.6}$$

where $|\eta,k\rangle_{\substack{\text{NSNS}\\ \text{RR}}}$ carries momentum in the $9-p$ Dirichlet directions.[8] These equations can be easily solved via coherent states

$$\begin{aligned}
|\eta,k\rangle_{\substack{\text{NSNS}\\ \text{RR}}} = \exp\Bigg(&\sum_{n=1}^\infty \frac{1}{n}\Bigg(-\sum_{\mu=0,\ldots,p} \alpha_{-n}^\mu \widetilde{\alpha_{-n}}^\mu + \sum_{i=p+1}^7 \alpha_{-n}^i \widetilde{\alpha_{-n}}^i\Bigg) + \\
&+ i\eta \sum_{r>0}\Bigg(-\sum_{\mu=0,\ldots,p} \psi_{-r}^\mu \widetilde{\psi_{-r}}^\mu + \sum_{i=p+1}^7 \psi_{-r}^i \widetilde{\psi_{-r}}^i\Bigg)\Bigg)\, |\eta,k\rangle^{(0)}_{\substack{\text{NSNS}\\ \text{RR}}}
\end{aligned} \tag{3.7}$$

using the left moving and right moving oscillators of the string. The Fock vacuum $|\eta,k\rangle^{(0)}_{\substack{\text{NSNS}\\ \text{RR}}}$ is unique in the NSNS sector and it is identical to the usual closed string vacuum. The RR vacuum is more complicated and we will now describe it in a way which will be useful in the orbifold construction. These vacua must also solve (3.6) for the zero modes. It turns out to be convenient to rewrite the zero mode equations

[6] We will a notation very similar to that in [35, 36].

[7] See appendix for conventions.

[8] Strictly speaking the Dp-brane is obtained only after analytically continuing back.

in the creation-annihilation basis of the SO(8) Clifford algebra $\Gamma^{a\pm} = \frac{1}{2}(\gamma^{2a} \pm i\gamma^{2a+1})$, which satisfy

$$\{\Gamma^{a+}, \Gamma^{b-}\} = \delta^{ab}\,, \qquad \left(\Gamma^{a+}\right)^2 = \left(\Gamma^{a-}\right)^2 = 0\,, \qquad a = 0, \ldots, 3\,. \tag{3.8}$$

For p odd the equations become

$$\begin{aligned} (\Gamma^{b-} + i\eta\widetilde{\Gamma}^{b-})|\eta, k\rangle^{(0)}_{\mathrm{RR}} &= 0\,, \qquad b = 0, \ldots, \frac{p-1}{2}\,, \\ (\Gamma^{c-} - i\eta\widetilde{\Gamma}^{c-})|\eta, k\rangle^{(0)}_{\mathrm{RR}} &= 0\,, \qquad c = \frac{p+1}{2}, \ldots, 3 \end{aligned} \tag{3.9}$$

and for p even

$$\begin{aligned} (\Gamma^{d-} + i\eta\widetilde{\Gamma}^{d-})|\eta, k\rangle^{(0)}_{\mathrm{RR}} &= 0\,, \qquad d = 0, \ldots, p/2 - 1\,, \\ (\Gamma^{p/2-} + i\eta\widetilde{\Gamma}^{p/2+})|\eta, k\rangle^{(0)}_{\mathrm{RR}} &= 0\,, \\ (\Gamma^{e-} - i\eta\widetilde{\Gamma}^{e-})|\eta, k\rangle^{(0)}_{\mathrm{RR}} &= 0\,, \qquad d = p/2 + 1, \ldots, 3\,. \end{aligned} \tag{3.10}$$

Hence the solution to these equations are

$$|\eta, k\rangle^{(0)}_{\mathrm{RR}} = \exp\left(i\eta(-\Gamma^{b+}\widetilde{\Gamma}^{b-} + \Gamma^{c+}\widetilde{\Gamma}^{c-})\right)|0, k\rangle_{\mathrm{RR}} \otimes \widetilde{|0, k\rangle}_{\mathrm{RR}} \qquad p \text{ odd}\,, \tag{3.11}$$

$$|\eta, k\rangle^{(0)}_{\mathrm{RR}} = \exp\left(i\eta(-\Gamma^{d+}\widetilde{\Gamma}^{d-} - \Gamma^{\frac{p}{2}+}\widetilde{\Gamma}^{p/2+} + \Gamma^{e+}\widetilde{\Gamma}^{e-})\right)|0, k\rangle_{\mathrm{RR}} \otimes \widetilde{|0, k\rangle}_{\mathrm{RR}} \qquad p \text{ even}\,.$$

The Fock vacua are defined such that

$$\Gamma^{a-}|0, k\rangle_{\mathrm{RR}} = 0\,, \qquad a = 0, \ldots, 3 \tag{3.12}$$

and

$$\begin{aligned} &\widetilde{\Gamma}^{a+}\widetilde{|0, k\rangle}_{\mathrm{RR}} = 0\,, \qquad a = 0, \ldots, 3\,, \qquad \text{For } p \text{ odd}\,, \\ &\widetilde{\Gamma}^{d+}\widetilde{|0, k\rangle}_{\mathrm{RR}} = 0\,, \qquad \widetilde{\Gamma}^{p/2-}\widetilde{|0, k\rangle}_{\mathrm{RR}} = 0\,, \\ &\widetilde{\Gamma}^{e+}\widetilde{|0, k\rangle}_{\mathrm{RR}} = 0 \qquad \text{For } p \text{ even}\,, \end{aligned} \tag{3.13}$$

where the indices are as in (3.9). These are easily described as vectors in the SO(8) weight lattice in the standard way. They are

$$|0, k\rangle_{\mathrm{RR}} \longrightarrow \left(-\frac{1}{2}^4\right) \tag{3.14}$$

and

$$\begin{aligned} \widetilde{|0, k\rangle}_{\mathrm{RR}} &\longrightarrow \left(\frac{1}{2}^4\right) \qquad \text{For } p \text{ odd}\,, \\ \widetilde{|0, k\rangle}_{\mathrm{RR}} &\longrightarrow \left(-\frac{1}{2}\frac{1}{2}^3\right) \qquad \text{For } p \text{ even}\,, \end{aligned} \tag{3.15}$$

which have opposite chirality.

We have constructed the Ishibashi states for the flat space-vacuum. We will now construct consistent boundary states. These states must be invariant under all the symmetries of the closed string CFT and must factorize via a modular transformation on the cylinder amplitude. That is, the open string one-loop cylinder amplitude with a given set of boundary conditions must agree with the answer that results from propagation in the closed string channel between boundary states

$$Z(t) = \int \frac{dt}{2t} \operatorname{tr} \left(\frac{1+(-1)^F}{2} e^{-2tH_o} \right) = \int dl \langle B| e^{-lH_c} |B\rangle \,, \tag{3.16}$$

such that the open string time t and the closed time l are related by $t = 1/2l$. The boundary states must satisfy the GSO projection of the underlying string theory they are embedded in. Therefore, they must satisfy

$$\begin{aligned} (-1)^F |B\rangle &= (-1)^{\widetilde{F}} |B\rangle = |B\rangle \qquad && \text{type IIB}\,, \\ (-1)^F |B\rangle &= -(-1)^{\widetilde{F}} |B\rangle = |B\rangle \qquad && \text{type IIA}\,. \end{aligned} \tag{3.17}$$

The GSO operators act on the fermion zero modes (Gamma matrices) as the chirality matrix so that $(-1)^F \Gamma^{a\pm} = -\Gamma^{a\pm}(-1)^F$ and $(-1)^F \psi_r = -\psi_r (-1)^F$ and similarly for the right movers. Furthermore, since $|\eta, k\rangle^{(0)}_{\rm NSNS}$ is odd under both $(-1)^F$ and $(-1)^{\widetilde{F}}$, it follows that the action on the Ishibashi states (3.7) is

$$(-1)^F |\eta, k\rangle_{\rm NSNS} = -|-\eta, k\rangle_{\rm NSNS}\,, \qquad (-1)^{\widetilde{F}} |\eta, k\rangle_{\rm NSNS} = -|-\eta, k\rangle_{\rm NSNS}\,, \tag{3.18}$$

so that the GSO invariant combination in the NSNS sector is

$$\frac{1}{\sqrt{2}} (|+\rangle_{\rm NSNS} - |-\rangle_{\rm NSNS})\,. \tag{3.19}$$

Using (3.7), (3.11) and (3.15) one gets for the RR sector

$$(-1)^F |\eta, k\rangle_{\rm RR} = |-\eta, k\rangle_{\rm RR} \tag{3.20}$$

and

$$\begin{aligned} (-1)^{\widetilde{F}} |\eta, k\rangle_{\rm RR} &= |-\eta, k\rangle_{\rm RR} \qquad && p \text{ odd}\,, \\ (-1)^{\widetilde{F}} |\eta, k\rangle_{\rm RR} &= -|-\eta, k\rangle_{\rm RR} \qquad && p \text{ even}\,. \end{aligned} \tag{3.21}$$

Therefore, invariance under $(-1)^F$ in the RR sector yields the following linear combination

$$\frac{1}{\sqrt{2}} (|+\rangle_{\rm RR} + |-\rangle_{\rm RR})\,. \tag{3.22}$$

Imposing the right moving GSO invariance of (3.17) restricts p to be odd for the type-IIB superstring and p even for the type-IIA superstring just as expected for supersymmetric branes.[9]

[9] One can write boundary states for any value of p for the type-II theories, but if they are not even(odd) for type-IIA (IIB) they do not carry Ramond-Ramond charge and are unstable due to the presence of a tachyon in the open string channel.

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We must further impose factorizability with the open string calculation. The open string hamiltonian is given by

$$H_o = \pi p^2 + \pi \sum_{\mu=0,1,\ldots,7} \left(\sum_{n=1}^{\infty} \alpha^{\mu}_{-n} \alpha^{\mu}_{n} + \sum_{r>0} r \psi^{\mu}_{-r} \psi^{\mu}_{r} \right) + \pi C_0 \,, \tag{3.23}$$

with C_0 zero in the Ramond sector and $-1/2$ in the NS sector. Therefore, the partition function for an open string with $p+1$ Neumann boundary conditions and $9-p$ Dirichlet boundary conditions in the light cone is

$$\begin{aligned} Z &= \int \frac{dt}{2t} \operatorname{tr} \left(\frac{1+(-1)^F}{2} e^{-2tH_0} \right) \\ &= \frac{V_{p+1}}{(2\pi)^{p+1}} \left(\frac{1}{2} \right)^{\frac{p+3}{2}} \int \frac{dt}{2t^{\frac{p+3}{2}}} \frac{1}{\eta(it)^8} \left(\left(\frac{\vartheta_3(0,it)}{\eta(it)} \right)^4 - \left(\frac{\vartheta_4(0,it)}{\eta(it)} \right)^4 - \left(\frac{\vartheta_2(0,it)}{\eta(it)} \right)^4 \right) . \end{aligned} \tag{3.24}$$

V_{p+1} is the volume of the Dp-brane, the first two terms correspond to tracing over the NS sector without and with the GSO insertion $(-1)^F$ and the last term is the trace over the R sector without any insertion. These can be conveniently written using ϑ functions

$$\begin{aligned} \eta(\tau) &= q^{1/24} \prod_{n=1}^{\infty} (1-q^n) \,, \\ \vartheta_1(\nu,\tau) &= 2 \exp\left(\frac{\pi i \tau}{4} \right) \sin(\pi\nu) \prod_{n=1}^{\infty} (1-q^n)(1-e^{2\pi i \nu} q^n)(1-e^{-2\pi i\nu} q^n) \,, \\ \vartheta_2(\nu,\tau) &= 2 \exp\left(\frac{\pi i \tau}{4} \right) \cos(\pi\nu) \prod_{n=1}^{\infty} (1-q^n)(1+e^{2\pi i \nu} q^n)(1+e^{-2\pi i\nu} q^n) \,, \\ \vartheta_3(\nu,\tau) &= \prod_{n=1}^{\infty} (1-q^n)(1+e^{2\pi i\nu} q^{n-1/2})(1+e^{-2\pi i\nu} q^{n-1/2}) \,, \\ \vartheta_4(\nu,\tau) &= \prod_{n=1}^{\infty} (1-q^n)(1-e^{2\pi i\nu} q^{n-1/2})(1-e^{-2\pi i\nu} q^{n-1/2}) \,, \end{aligned} \tag{3.25}$$

where $q = e^{2\pi i \tau}$.

In order to compare with the closed string channel, we must rewrite (3.24) in closed string time $l = 1/2t$. Using modular properties of ϑ-functions

$$\begin{aligned} \eta(\tau) &= (-i\tau)^{-1/2} \eta\left(-\frac{1}{\tau} \right) , \\ \vartheta_1(\nu,\tau) &= i(-i\tau)^{-1/2} e^{-\pi i \nu^2/\tau} \vartheta_1 \left(\frac{\nu}{\tau}, \frac{-1}{\tau} \right) , \\ \vartheta_2(\nu,\tau) &= (-i\tau)^{-1/2} e^{-\pi i\nu^2/\tau} \vartheta_4 \left(\frac{\nu}{\tau}, \frac{-1}{\tau} \right) , \end{aligned}$$

$$\begin{aligned}\vartheta_3(\nu,\tau) &= (-i\tau)^{-1/2} e^{-\pi i \nu^2/\tau} \vartheta_3\left(\frac{\nu}{\tau}, \frac{-1}{\tau}\right),\\ \vartheta_4(\nu,\tau) &= (-i\tau)^{-1/2} e^{-\pi i \nu^2/\tau} \vartheta_2\left(\frac{\nu}{\tau}, \frac{-1}{\tau}\right)\end{aligned} \tag{3.26}$$

one gets

$$Z = \frac{V_{p+1}}{(2\pi)^{p+1}} \frac{1}{64} \int \frac{dl}{l^{\frac{9-p}{2}}} \frac{1}{\eta(2il)^8} \left(\left(\frac{\vartheta_3(0,2il)}{\eta(2il)}\right)^4 - \left(\frac{\vartheta_4(0,2il)}{\eta(2il)}\right)^4 - \right. \\ \left. - \left(\frac{\vartheta_2(0,2il)}{\eta(2il)}\right)^4 \right). \tag{3.27}$$

It is clear that in order to reproduce this expression using boundary states (3.16), that we must put them in a position eigenstate[10]

$$|\eta\rangle_{\rm NSNS} = \mathcal{N} \int d^{9-p}k\ |\eta, k\rangle_{\rm NSNS}\,, \qquad |\eta\rangle_{\rm RR} = 4i\mathcal{N} \int d^{9-p}k\ |\eta, k\rangle_{\rm RR}\,, \tag{3.28}$$

so that the powers of closed string time in (3.27) match.

There is a heuristic way of anticipating which set of "matrix elements" of boundary state components result in the various terms in (3.27). We can think of the R sector of the open string as a twisted sector of the NS sector when we go around σ. Moreover, the insertion of $(-1)^F$ in the trace is a twist in the t direction. Now, since the roles of t and σ are exchanged in the closed string channel, we can make the following identifications:[11]

$$\begin{aligned}\int \frac{dt}{2t}\, {\rm tr}_{\rm NS}(e^{-2tH_0}) &= \int dl\ {}_{\rm NSNS}\langle +|e^{-lH_c}|+\rangle_{\rm NSNS}\\ &= \int dl\ {}_{\rm NSNS}\langle -|e^{-lH_c}|-\rangle_{\rm NSNS}\,,\\ \int \frac{dt}{2t}\, {\rm tr}_{\rm NS}((-1)^F e^{-2tH_0}) &= \int dl\ {}_{\rm RR}\langle +|e^{-lH_c}|+\rangle_{\rm RR} = \int dl\ {}_{\rm RR}\langle -|e^{-lH_c}|-\rangle_{\rm RR}\,,\\ \int \frac{dt}{2t}\, {\rm tr}_{R}(e^{-2tH_0}) &= \int dl\ {}_{\rm NSNS}\langle +|e^{-lH_c}|-\rangle_{\rm NSNS}\\ &= \int dl\ {}_{\rm NSNS}\langle -|e^{-lH_c}|+\rangle_{\rm NSNS}\end{aligned} \tag{3.29}$$

and all other "matrix elements" vanish. One can easily perform the calculation of these matrix elements using the explicit expressions for the boundary states we wrote down and using as the closed string hamiltonian

$$H_c = \pi p^2 + 2\pi \sum_{\mu=0,1,\ldots,7} \left(\sum_{n=1}^{\infty} \alpha^\mu_{-n}\alpha^\mu_n + \sum_{r>0} r\psi^\mu_{-r}\psi^\mu_r \right) + 2\pi C_0\,, \tag{3.30}$$

[10] Following [35], we have chosen to define the corresponding Ramond-Ramond bra vectors without conjugating the i.

[11] The normalization constant $\mathcal{N}$ is found by requiring these expressions to be equal.

with C_0 zero in the RR sector and -1 in the NSNS sector. One gets the open string answer if we take our boundary states representing a D-brane to be[12]

$$|B\rangle = \frac{1}{2}\left(|+\rangle_{\mathrm{NSNS}} - |-\rangle_{\mathrm{NSNS}} + |+\rangle_{\mathrm{RR}} + |-\rangle_{\mathrm{RR}}\right). \tag{3.31}$$

and the normalization constant in (3.28) is $\mathcal{N}^2 = \frac{V_{p+1}}{(2\pi)^{p+1}}\frac{1}{32}$.

After introducing the necessary ingredients we are now ready to discuss the realization of fractional branes in orbifold backgrounds.

4. Fractional branes and boundary states

We will use the open string interpretation of the cylinder amplitude to classify the consistent set of supersymmetric boundary states corresponding to D-branes at a $\mathbb{C}^3/\Gamma$ singularity. We will analyze branes that are point-like in the orbifold direction and in the transverse space.

An open string in this background suffers identifications which follow from the action of Γ. Γ acts both on the coordinates along the orbifold and on the end-points of the string. The action on the coordinates is given by the action of Γ on the $\mathbf{3}$ of SU(3). The action on the Chan-Paton factors is determined by choosing a representation of Γ as in the gauge theory discussion. Therefore, a general one-loop open string amplitude can be obtained from a set of n cylinder amplitudes obtained by acting with the a-th irreducible representation γ_a, $a = 1,\dots,n$ on the Chan-Paton factors. Therefore, we expect to be able to construct as many basic consistent boundary states of the $\mathbb{C}^3/\Gamma$ background as the number of irreducible representations, n, that Γ admits. The most general boundary state will admit an expansion in terms of these basic ones.

The a-th open string cylinder amplitude is given by

$$Z^a = \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\int\frac{dt}{2t}\,\mathrm{tr}\left(g\frac{1+(-1)^F}{2}e^{-2tH_o}\right), \tag{4.1}$$

where H_o is the open string hamiltonian. This amplitude can be written as

$$Z^a = \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\chi_a^2(g)Z_g(\widetilde{q}), \tag{4.2}$$

where χ_a is the character of the a-th irreducible representation of Γ and

$$Z_g = \int\frac{dt}{2t}\,\mathrm{tr}\left(g\frac{1+(-1)^F}{2}e^{-2tH_o}\right). \tag{4.3}$$

[12]We can easily obtain the expression for an anti-D-brane by changing the sign of the RR contribution to the boundary state.

As mentioned above, these open string amplitudes should have an interpretation as propagation between boundary states. In this case proper factorization demands that

$$\int dl\, {}_a\langle B|e^{-lH_c}|B\rangle_a = Z^a\,. \tag{4.4}$$

We will see in different examples how this identification works and explicitly construct the boundary states $|B\rangle_a$.

Since we are interested in branes that look point-like in the orbifold direction and in the transverse space, the boundary conditions that we will impose on the closed string worldsheet fields are the ones in (3.6) with $p = 0$. A proper linear combination of solutions has to be formed such that the boundary states are GSO invariant, Γ invariant and factorize properly in the open string channel (4.4). The presence of the orbifold forces us to consider solutions to (3.6) in all twisted sectors of the closed string. Therefore, the full boundary state associated with the a-th irreducible representation in the open string channel is formed as a linear combination of boundary states in the n string sectors

$$|B\rangle_a = \frac{1}{\sqrt{n}}\sum_{m=0}^{n-1}|B,m\rangle_a\,. \tag{4.5}$$

We shall see that the matrix elements involving $|B,m\rangle_a$ yield the open string answer when the corresponding group element is inserted in the open string trace. These boundary states carry charges under twisted sector vertex operators $|s\rangle$ as measured by the overlap $\langle s|B\rangle_a$. In particular they carry charge under twisted RR fields and as we will see with precisely the correct value as the corresponding fractional brane in the probe theory approach. We will now consider several examples including orbifolds with isolated singularities, non-isolated singularities and discrete torsion.

4.1 $\mathbb{C}^3/\mathbb{Z}_N$ orbifold

We will first compute the open string cylinder amplitudes that the boundary states must reproduce. The action of the $\mathbb{Z}_N$ generator g on the worldsheet fields is implemented by

$$g = \exp^{\frac{2\pi i}{N}(a_1 s_1 + a_2 s_2 + a_3 s_3)}\,, \tag{4.6}$$

where (s_1, s_2, s_3) are vectors in the SO(6) weight lattice corresponding to $\mathbb{C}^3$. It is therefore useful to introduce three complex coordinates and their complex conjugate which describe the orbifold

$$\begin{aligned} Z^i &= \frac{1}{\sqrt{2}}(X^{2i} + iX^{2i+1})\,, \qquad \bar{Z}^i = \frac{1}{\sqrt{2}}(X^{2i} - iX^{2i+1})\,, \qquad i = 1,2,3 \\ \lambda^i &= \frac{1}{\sqrt{2}}(\psi^{2i} + i\psi^{2i+1})\,, \qquad \bar{\lambda}^i = \frac{1}{\sqrt{2}}(\psi^{2i} - i\psi^{2i+1})\,, \end{aligned} \tag{4.7}$$

so that[13]

$$\begin{aligned} &Z^i \longrightarrow \alpha^{a_i} Z^i\,, \qquad \bar{Z}^i \longrightarrow \alpha^{-a_i} \bar{Z}^i\,, \qquad a_1+a_2+a_3=0\ (N)\\ &\lambda^i \longrightarrow \alpha^{a_i}\lambda^i\,, \qquad \bar{\lambda}^i \longrightarrow \alpha^{-a_i}\bar{\lambda}^i\,, \end{aligned} \tag{4.8}$$

where $\alpha = e^{2\pi i/N}$ and the action on the open string vacua yield

$$g\cdot|0\rangle_{\rm NS} = |0\rangle_{\rm NS}\,, \qquad g\cdot|0\rangle_R = 16\prod_{i=1}^{3}\cos(\pi\nu_i)|0\rangle_R\,, \tag{4.9}$$

where $\nu_i = a_i/N$.

Performing the trace in (4.3) yields

$$\begin{aligned} Z_{g^0} = \frac{L}{2\pi}\frac{1}{2\sqrt{2}}\int\frac{dt}{2t^{3/2}}\frac{1}{\eta(it)^8}\Bigg(\left(\frac{\vartheta_3(0,it)}{\eta(it)}\right)^4 - \left(\frac{\vartheta_4(0,it)}{\eta(it)}\right)^4 -\\ - \left(\frac{\vartheta_2(0,it)}{\eta(it)}\right)^4\Bigg)\,, \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} Z_{g^m} &= \frac{L}{2\pi}\frac{4}{\sqrt{2}}\int\frac{dt}{2t^{3/2}}\frac{1}{\eta(it)^3}\times\\ &\times\prod_{i=1}^{3}\frac{\sin(\pi m\nu_i)}{\vartheta_1(m\nu_i,it)}\Bigg(\vartheta_3(0,it)\prod_{i=1}^{3}\vartheta_3(m\nu_i,it) - \vartheta_4(0,it)\prod_{i=1}^{3}\vartheta_4(m\nu_i,it) -\\ &- \vartheta_2(0,it)\prod_{i=1}^{3}\vartheta_2(m\nu_i,it)\Bigg)\,, \end{aligned} \tag{4.11}$$

where $m = 1,\ldots,N-1$ and $L/2\pi$ comes from integration over x^0. Worldsheet duality relates this open string amplitude to the propagation in closed string time $l = 1/2t$ between boundary states. Therefore, we must rewrite (4.11) in closed string time using modular properties of ϑ functions. This yields the following expressions:

$$\begin{aligned} Z_{g^0} = \frac{L}{2\pi}\frac{1}{64}\int\frac{dl}{l^{9/2}}\frac{1}{\eta(2il)^8}\Bigg(\left(\frac{\vartheta_3(0,2il)}{\eta(2il)}\right)^4 - \left(\frac{\vartheta_4(0,2il)}{\eta(2il)}\right)^4 -\\ - \left(\frac{\vartheta_2(0,2il)}{\eta(2il)}\right)^4\Bigg)\,, \end{aligned} \tag{4.12}$$

[13] See appendix for conventions.

and

$$Z_{g^m} = i\frac{L}{2\pi}\int \frac{dl}{l^{3/2}}\frac{1}{\eta(2il)^3}\prod_{i=1}^{3}\frac{\sin(\pi m\nu_i)}{\vartheta_1(-2im\nu_i l, 2il)}\times$$
$$\times\left(\vartheta_3(0,2il)\prod_{i=1}^{3}\vartheta_3(-2im\nu_i l,2il) - \vartheta_4(0,2il)\prod_{i=1}^{3}\vartheta_4(-2im\nu_i l,2il) - \right.$$
$$\left. - \vartheta_2(0,2il)\prod_{i=1}^{3}\vartheta_2(-2im\nu_i l,2il)\right). \tag{4.13}$$

The expressions we will find next for the boundary states will reproduce this one and will thus constitute consistent D-brane states.

4.2 Boundary states in $\mathbb{C}^3/\mathbb{Z}_N$

We will now show that the term in the open string partition function with g^m inserted in the trace (4.13) is reproduced by taking the solutions of (3.6) in the m-th twisted sector of the closed string and constructing the corresponding coherent states in the RR and NSNS sector.

The untwisted sector solutions to (3.6) are identical to those in flat space just discussed. However, we must also make sure that the boundary state we write down is $\mathbb{Z}_N$ invariant. It is therefore convenient to write the Ishibashi states using the complex coordinates (4.8). Then,[14]

$$|\eta,k,0\rangle_{\substack{\mathrm{NSNS}\\ \mathrm{RR}}} = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\left(-\alpha^0_{-n}\widetilde{\alpha_{-n}}^{0} + \alpha^1_{-n}\widetilde{\alpha_{-n}}^{1} + \sum_{i=1}^{3}\beta^i_{-n}\widetilde{\bar\beta}^i_{-n} + \bar\beta^i_{-n}\widetilde{\beta}^i_{-n}\right) + \right.$$
$$\left. + i\eta\sum_{r>0}\left(-\psi^0_{-r}\widetilde{\psi}^0_{-r} + \psi^1_{-r}\widetilde{\psi}^1_{-r} + \sum_{i=1}^{3}\lambda^i_{-r}\widetilde{\bar\lambda}^i_{-r} + \bar\lambda^i_{-r}\widetilde{\lambda}^i_{-r}\right)\right)\times$$
$$\times|\eta,k,0\rangle^{(0)}_{\substack{\mathrm{NSNS}\\ \mathrm{RR}}}. \tag{4.14}$$

$\beta, \bar\beta, \lambda, \bar\lambda$ are the left moving oscillators of the complex worldsheet fields and similarly for the right movers.[15] Written in this form it is clear that the exponential is neutral under the action of $\mathbb{Z}_N$ since each oscillator appears multiplying a complex conjugate one. Furthermore, using (4.6) and the explicit expressions for the vacua we see that in fact the full set of Ishibashi states (4.14) are $\mathbb{Z}_N$ invariant in the untwisted sector. Therefore the expression reproducing the open string result without any $\mathbb{Z}_N$ element in the trace (4.12) is (3.31) with

$$\mathcal{N}_0^{a2} = \frac{L}{2\pi}\frac{1}{32}. \tag{4.15}$$

[14] We introduce another label for the boundary states which describes the sector of closed string oscillators we use.

[15] See appendix for mode expansions.

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In closed string theory the string can be closed up to an action of g^m. This sector of the string is usually referred to the m-twisted sector. It is easy to see that in a twisted sector the modding of the oscillators along the orbifold directions get shifted. In the $\mathbb{C}^3/\mathbb{Z}_N$ example it is easy to see from the mode expansions in the appendix that the modified moddings in the m-th twisted sector are[16]

$$
\begin{array}{cccc}
\beta^i_{n+m\nu_i} & \widetilde{\beta}^i_{n-m\nu_i} & \bar{\beta}^i_{n-m\nu_i} & \widetilde{\bar{\beta}}^i_{n+m\nu_i} \\
\lambda^i_{r+m\nu_i} & \widetilde{\lambda}^i_{r-m\nu_i} & \bar{\lambda}^i_{r-m\nu_i} & \widetilde{\bar{\lambda}}^i_{r+m\nu_i}
\end{array}
\tag{4.16}
$$

which satisfy

$$
\begin{aligned}
[\beta^i_{n+m\nu_i}, \bar{\beta}^j_{p-m\nu_i}] = [\widetilde{\beta}^i_{n-m\nu_i}, \widetilde{\bar{\alpha}}^j_{p+m\nu_i}] &= (n+m\nu_i)\delta_{n+p}\delta_{ij}\,, \\
\{\lambda^i_{r+m\nu_i}, \bar{\lambda}^j_{s-m\nu_i}\} = \{\widetilde{\lambda}^i_{r-m\nu_i}, \widetilde{\bar{\lambda}}^j_{s+m\nu_i}\} &= \delta_{r+s}\delta_{ij}
\end{aligned}
\tag{4.17}
$$

with the rest of (anti)commutators vanishing. Moreover, the orbifold action projects out the bosonic and fermionic zero modes along the orbifold directions since twisted sector states are stuck at the origin of the singularity. Technically, we have to perform the same construction as in flat space. The boundary conditions to be solved are those in (3.6) with the shifted oscillators in the orbifold directions

$$
\begin{aligned}
(\beta^i_{n+m\nu_i} - \widetilde{\beta}^i_{-n-m\nu_i})|\eta,k,m\rangle_{\substack{\mathrm{NSNS}\\ \mathrm{RR}}} &= 0\,, \\
(\bar{\beta}^i_{n-m\nu_i} - \widetilde{\bar{\beta}}^i_{-n+m\nu_i})|\eta,k,m\rangle_{\substack{\mathrm{NSNS}\\ \mathrm{RR}}} &= 0\,, \\
(\lambda^i_{r+m\nu_i} - i\eta\widetilde{\lambda}^i_{-r-m\nu_i})|\eta,k,m\rangle_{\substack{\mathrm{NSNS}\\ \mathrm{RR}}} &= 0\,, \\
(\bar{\lambda}^i_{r-m\nu_i} - i\eta\widetilde{\bar{\lambda}}^i_{-r+m\nu_i})|\eta,k,m\rangle_{\substack{\mathrm{NSNS}\\ \mathrm{RR}}} &= 0\,.
\end{aligned}
\tag{4.18}
$$

Therefore, the Ishibashi states in the m-th twisted sector are given by

$$
\begin{aligned}
&|\eta,k,m\rangle_{\substack{\mathrm{NSNS}\\ \mathrm{RR}}} = \\
&= \exp\Bigg(\sum_{n=1}^{\infty}\frac{1}{n}(-\alpha^0_{-n}\widetilde{\alpha_{-n}}^0 + \alpha^1_{-n}\widetilde{\alpha_{-n}}^1) + \sum_{i=1}^{3}\frac{1}{n-m\nu_i}\beta^i_{-n+m\nu_i}\widetilde{\bar{\beta}}^i_{-n+m\nu_i} + \\
&\qquad + \frac{1}{n+m\nu_i}\bar{\beta}^i_{-n-m\nu_i}\widetilde{\beta}^i_{-n-m\nu_i} + \\
&\qquad + i\eta\sum_{r>0}\Bigg(-\psi^0_{-r}\widetilde{\psi}^0_{-r} + \psi^1_{-r}\widetilde{\psi}^1_{-r} + \sum_{i=1}^{3}\lambda^i_{-r+m\nu_i}\widetilde{\bar{\lambda}}^i_{-r+m\nu_i} + \\
&\qquad\qquad + \bar{\lambda}^i_{-r-m\nu_i}\widetilde{\lambda}^i_{-r-m\nu_i}\Bigg)\Bigg)|\eta,k,m\rangle^{(0)}_{\substack{\mathrm{NSNS}\\ \mathrm{RR}}}\,,
\end{aligned}
\tag{4.19}
$$

[16]Since we are interested in isolated singularities we consider N odd. We will consider examples of non-isolated singularities in an upcoming section.

where $|\eta, k, m\rangle_{\rm RR}^{(0)}$ is constructed as before but only with the zero modes of the transverse coordinates x^0, x^1. Therefore, the GSO, $\mathbb{Z}_N$ invariant boundary state in the m-th twisted sector is[17]

$$
\begin{aligned}
|B, m\rangle^a &= \frac{1}{2}\Big(|+, m\rangle^a_{\rm NSNS} - |-, m\rangle^a_{\rm NSNS} + |+, m\rangle^a_{\rm RR} + |-, m\rangle^a_{\rm RR}\Big) \\
m &= 1, \ldots, N-1\,.
\end{aligned}
\tag{4.20}
$$

However, since the orbifold removes the zero modes along the orbifold, now the Fourier transform is only in the transverse directions

$$
\begin{aligned}
|\eta, m\rangle^a_{\rm NSNS} &= \mathcal{N}^a_m \int d^3k\ |\eta, m, k\rangle_{\rm NSNS}\,, \\
|\eta, m\rangle_{\rm RR} &= \sqrt{2} i \mathcal{N}^a_m \int d^3k\ |\eta, m, k\rangle_{\rm RR}\,.
\end{aligned}
\tag{4.21}
$$

Using the explicit form of these boundary states, we will now see that the different matrix elements in (4.20) yield the open string answer (4.13). Therefore,

$$
\begin{aligned}
&\int dl\, {}^a_{\rm NSNS}\langle +, m| e^{-lH_c} |+, m\rangle^a_{\rm NSNS} = \int dl\, {}^a_{\rm NSNS}\langle -, m| e^{-lH_c} |-, m\rangle^a_{\rm NSNS} \\
&\quad = i \mathcal{N}^{a2}_m \int dl \frac{1}{l^{3/2}} \prod_{i=1}^3 \frac{1}{\vartheta_1(-2im\nu_i l, 2il)} \vartheta_3(0, 2il) \prod_{i=1}^3 \vartheta_3(-2im\nu_i l, 2il)\,, \\
&\int dl\, {}^a_{\rm RR}\langle +, m| e^{-lH_c} |+, m\rangle^a_{\rm RR} = \int dl\, {}^a_{\rm RR}\langle -, m| e^{-lH_c} |-, m\rangle^a_{\rm RR} \\
&\quad = -i \mathcal{N}^{a2}_m \int dl \frac{1}{l^{3/2}} \prod_{i=1}^3 \frac{1}{\vartheta_1(-2im\nu_i l, 2il)} \vartheta_4(0, 2il) \prod_{i=1}^3 \vartheta_4(-2im\nu_i l, 2il)\,, \\
&\int dl\, {}^a_{\rm NSNS}\langle +, m| e^{-lH_c} |-, m\rangle^a_{\rm NSNS} = \int dl\, {}^a_{\rm NSNS}\langle -, m| e^{-lH_c} |+, m\rangle^a_{\rm NSNS} \\
&\quad = i \mathcal{N}^{a2}_m \int dl \frac{1}{l^{3/2}} \prod_{i=1}^3 \frac{1}{\vartheta_1(-2im\nu_i l, 2il)} \vartheta_2(0, 2il) \prod_{i=1}^3 \vartheta_2(-2im\nu_i l, 2il)
\end{aligned}
\tag{4.22}
$$

and all other "matrix elements" vanish.

Thus, the m-twisted sector contribution to the boundary state reproduces the open string amplitude with g^m inserted in the trace if we choose

$$
\mathcal{N}^{a2}_m = \frac{L}{2\pi} \frac{\chi^{a2}(g^m)}{4} 8 \prod_{i=1}^3 \sin(\pi m \nu_i)\,. \tag{4.23}
$$

We have therefore constructed a consistent boundary state for each irreducible representation of $\mathbb{Z}_N$, as expected. The factors of sin in (4.23) might seem awkward,

[17]The GSO operator acts the same as usual on oscillators and as the chirality matrix on the zero modes.

but in fact they are expected. Even though the orbifold action leaves fixed only the origin, in order to get a modular invariant closed string partition function, we must multiply the contribution from a given twisted sector by the fixed point degeneracy that the orbifold would have if it were toroidal. Now, for toroidal orbifolds, the number of fixed of the group element g^m is given by Lefschetz fixed-point theorem

$$\det(1-g^m) = 64\prod_{i=1}^{3}\sin^2(\pi m\nu_i)\,, \tag{4.24}$$

as can be shown using the defining action (4.6), and (4.23) contains the square root of this.

The charges carried by the boundary state can be computed by inserting the corresponding vertex operator. Thus, the charge vector of these states is made out of untwisted D0-brane charge and the twisted RR charges. The state that represents a D0-brane is the one made with the regular representation. Since the character of the regular representation is non-trivial only for the identity element, it is clear that the corresponding boundary state is charged only under the untwisted RR one-form as expected from the probe theory considerations. The D0-brane charge of the above states built on an irreducible representation is

$$Q_0^a = \frac{1}{N}\,, \qquad a = 1,\ldots,N\,. \tag{4.25}$$

These boundary states also carry charges under the RR twisted sectors. The charge under the m-th such sector can be easily computed to be

$$Q_m^a = \frac{\chi^a(g^m)}{N}\,, \qquad a = 1,\ldots,N\,, \tag{4.26}$$

as expected from the disk computation.

We have constructed all the boundary states corresponding to fractional branes for this orbifold. Any other supersymmetric D-brane state can be constructed out of these basic ones. These states can be identified in the large volume limit of the orbifold CFT moduli space with D-branes wrapping supersymmetric cycles.

4.3 $\mathbb{C}^3/\mathbb{Z}_N\times\mathbb{Z}_N$ orbifold

The orbifold group is generated by two elements g_1 and g_2, so that an arbitrary group element $g_1^p g_2^q$ acts as

$$\begin{aligned} &Z^1\longrightarrow\alpha^p Z^1\,, &&Z^2\longrightarrow\alpha^q Z^2\,, &&Z^3\longrightarrow\alpha^{-(p+q)}Z^3\,, &&p,q=0,\ldots,N-1\,,\\ &\psi^1\longrightarrow\alpha^p\psi^1\,, &&\psi^2\longrightarrow\alpha^q\psi^2\,, &&\psi^3\longrightarrow\alpha^{-(p+q)}\psi^3\,, && \end{aligned} \tag{4.27}$$

where $\alpha = e^{2\pi i/N}$ and the action on the vacua is

$$\begin{aligned} g\cdot|0\rangle_{\rm NS} &= |0>_{\rm NS}\,,\\ g\cdot|0\rangle_R &= 16\cos\left(\frac{\pi p}{N}\right)\cos\left(\frac{\pi q}{N}\right)\cos\left(\frac{\pi(p+q)}{N}\right)|0\rangle_R\,. \end{aligned} \tag{4.28}$$

This is an isolated singularity. There are $(N-1)(N-2)$ elements of this group that fix the origin. The contribution to the partition function with these group elements or the identity in the trace (4.3) is identical to that in the previous example (4.10) and (4.11) with the substitutions $k\nu_1 \to p$, $k\nu_2 \to q$ and $k\nu_3 \to -(p+q)$. Therefore the modular transform is (4.12) and (4.13).

There are $3(N-1)$ group elements that fix a line. The contribution of these group elements to the open string partition function can be accounted for by adding three copies of

$$\begin{aligned} Z_g(it) = \frac{L}{2\pi}\frac{2}{\sqrt{2}}\int \frac{dt}{2t^{3/2}}\frac{1}{\eta(it)^6}\frac{\sin(\pi p/N)^2}{\vartheta_1(p/N,it)^2} \times \\ \times \left(\vartheta_3(0,it)^2\vartheta_3\left(\frac{p}{N},it\right)^2 - \vartheta_4(0,it)^2\vartheta_4\left(\frac{p}{N},it\right)^2 - \right. \\ \left. - \vartheta_2(0,it)^2\vartheta_2\left(\frac{p}{N},it\right)^2\right), \end{aligned} \tag{4.29}$$

where $p = 1,\ldots,N-1$. The modular transform of this expression that we must reproduce from boundary states is

$$\begin{aligned} Z_g(2il) = -\frac{L}{2\pi}\frac{1}{4}\int \frac{dl}{l^{5/2}}\frac{1}{\eta(2il)^6}\frac{\sin(\pi p/N)^2}{\vartheta_1(-2ipl/N,2il)^2} \times \\ \times \left(\vartheta_3(0,2il)^2\vartheta_3\left(\frac{-2ipl}{N},2il\right)^2 - \vartheta_4(0,2il)^2\vartheta_4\left(\frac{-2ipl}{N},2il\right)^2 - \right. \\ \left. - \vartheta_2(0,2il)^2\vartheta_2\left(\frac{-2ipl}{N},2il\right)^2\right). \end{aligned} \tag{4.30}$$

4.4 Boundary states in $\mathbb{C}^3/\mathbb{Z}_N \times \mathbb{Z}_N$

The boundary state is again a sum over all twisted sectors

$$|B\rangle_a = \frac{1}{N}\sum_{m=0}^{N^2-1} |B,m\rangle_a\,. \tag{4.31}$$

The sectors associated with group elements fixing a point can be obtained from the previous example (4.20) and (4.23) with the above mentioned substitutions. The sectors corresponding to group elements fixing a line can be constructed as if we were dealing with a $\mathbb{C}^2/\mathbb{Z}_N$ orbifold singularity. There are extra zero modes for these so that

$$\begin{aligned} |\eta,m\rangle^a_{\rm NSNS} &= \mathcal{N}^a_m \int d^5k\ |\eta,m,k\rangle_{\rm NSNS}\,, \\ |\eta,m\rangle_{\rm RR} &= 2i\mathcal{N}^a_m \int d^5k\ |\eta,m,k\rangle_{\rm RR}\,, \end{aligned} \tag{4.32}$$

with the obvious modifications in (4.19). Factorization with the open string requires

$$\mathcal{N}_g^a = \frac{L}{2\pi}\frac{\chi^{a2}(g)}{8} 4\sin^2\left(\frac{\pi p}{N}\right), \tag{4.33}$$

where the fixed point degeneracy appears as expected. In this example the untwisted D0-brane charge is

$$Q_0^a = \frac{1}{N^2}, \qquad a = 1, \ldots, N^2 \tag{4.34}$$

and the charge under the m-th twisted RR field

$$Q_m^a = \frac{\chi^a(g^m)}{N^2}, \qquad a = 1, \ldots, N^2 . \tag{4.35}$$

4.5 $\mathbb{C}^3/\mathbb{Z}_N \times \mathbb{Z}_N$ with discrete torsion

It is easy to construct the boundary states describing fractional branes for this vacuum. Technically, the only difference is the use of projective representations, and therefore the appearance of modified characters in the open string computation. Once the discrete torsion cocycles are chosen — $H^2(\mathbb{Z}_N \times \mathbb{Z}_N, \mathrm{U}(1)) \simeq \mathbb{Z}_N$, so that there are $N-1$ non-trivial cohomology classes — we can construct exactly as in the previous example boundary states corresponding to fractional branes. However, their number is the number of irreducible representations in the cocycle class of the discrete torsion phases. Within each cocycle class there is a unique N dimensional irreducible projective representation so that there is a unique fractional brane. Its charges are those in (4.34) and (4.35) for $a = 1$ with the modified character of the projective representation.

5. Geometric interpretation

The construction of BPS D-particle states has been so far restricted to exactly solvable orbifold theories, where the boundary state approach proved to be a powerful tool. However, orbifold theories admit exactly marginal deformations which give rise to a moduli space. This fact leads to a natural question, namely how does the BPS spectrum behave under these perturbations? Although a precise answer may depend on the particular aspects of each model, we will outline the general expected behavior and establish connections to singularity theory and McKay correspondence. In the next section, we will treat the case of $\mathbb{C}^3/\mathbb{Z}_3$ in detail, as in illustration of the principles outlined below. Since the effect of marginal deformations is very different in theories with and without discrete torsion, we start the discussion with conventional orbifolds. The case with discrete torsion will be considered at the end of the section.

In the absence of discrete torsion, the marginal operators correspond to blow-up modes of the singularity, parameterizing the Kähler moduli space of the resolved space [21]. Moreover, by adjusting the coefficients of the perturbations, we can eventually reach a region in the moduli space where the resulting exceptional cycles are very large and classical geometry is a good approximate description of the theory. Assuming that the BPS spectrum can be continuously deformed to this region with no jumping phenomena, we expect the orbifold states to be realized as D-branes wrapped on supersymmetric cycles in a smooth space-time background. Note that in this regime, there is no exact conformal field theory description, but the supergravity approximation is valid. Therefore we would obtain two different realizations of the same spectrum of BPS states. The assumption that jumping phenomena are absent is automatically satisfied in space-time theories with sixteen supercharges. In theories with eight supercharges, where this phenomenon is present, we simply assume that there is a path connecting the orbifold point to the large radius limit that avoids the curves of marginal stability.

For isolated singularities, carrying out this program is in fact equivalent to a physical realization of the celebrated McKay correspondence [13]–[20], as we now explain. Loosely, we can think of supersymmetric states in the large radius limit as D-branes wrapping supersymmetric cycles in space-time. The homology of the resolved space is even and generated by holomorphic cycles, which are supersymmetric. At the same time, the orbifold boundary states are in one-to-one correspondence with the irreducible representations of the orbifold group. Therefore, the deformation of BPS states effectively gives a map between the irreducible representations of the orbifold group and homology classes of the resolution. This would be a physical realization of the McKay correspondence.

A more precise formulation[18] can be achieved by interpreting the D0-brane states as K-theory classes [38, 6]. In the present context, D-brane states in an orbifold X/Γ are classified by the equivariant K-theory $K_\Gamma(X)$ with compact support of the covering space [6], [39]–[42]. A standard result [43] shows that $K_\Gamma(X)$ is isomorphic to the representation group $R(\Gamma)$ of the discrete group Γ generated by its irreducible representations. As mentioned before, this is in agreement with the boundary state construction. In the large radius limit, the D-brane states are classified similarly by a K-theory group supported on the exceptional divisor D of the resolved space $\widetilde{X}$. More precisely, the relevant K group $K_0(\widetilde{X})$ can be defined as the Grothendieck group of bounded complexes of algebraic vector bundles supported on the exceptional locus[19] [43, 44]. The relevance of this K-theory group in the context of D-branes on orbifolds and more general algebraic varieties has been discussed in [41, 45]. It can also be showed that $K_0(\widetilde{X})$ is isomorphic to the usual Grothendieck group $K(D)$ of coherent sheaves of the exceptional divisor D. When D is smooth, $K(D)$ is generated

[18]This paragraph is somewhat abstract and it is not essential for understanding the rest of the paper.

[19]This construction can be realized both in algebraic and topological setting, resulting in general in different objects. Since we are ultimately interested in BPS configurations, we will consider here the algebraic approach.

by classes of vector bundles. In this framework, the deformation of BPS states will give a map between the K-theory groups $K_\Gamma(X)$ and $K_0(\widetilde{X})$. This is a more precise formulation of the McKay correspondence [14, 15, 20].

In practice, deforming the spectrum of BPS states along the moduli space can be quite difficult, especially in theories with eight supercharges which exhibit quantum corrections. The $\mathbb{C}^3/\mathbb{Z}_3$ example studied in the next section illustrates the complexity of the problem.

For technical reasons, the above discussion has been restricted to isolated singularities. However, as detailed in section 3.3, the boundary state construction works as well for non-isolated singularities. We have showed that the fractional branes are again classified by the irreducible representations of the orbifold group. Therefore, in the $\mathbb{C}^3/\mathbb{Z}_N \times \mathbb{Z}_N$ orbifold treated there, we have (N^2-1) independent boundary states. The number of homology cycles of the exceptional locus can be computed using either conformal field theory techniques or toric methods. The abstract CFT Hodge diamond is [31]

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & \frac{(N+4)(N-1)}{2} & & 0 & \\
0 & & 0 & & 0 & & 0 \\
 & 0 & & \frac{(N+4)(N-1)}{2} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 0 & & &
\end{array} \tag{5.1}$$

Therefore, there seems to be a mismatch with the number of irreducible representations. A more careful analysis[20] shows that the Hodge diamond corresponding to normalizable cohomology classes or, by Poincaré duality, to compact cycles, is

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & \frac{(N-2)(N-1)}{2} & & 0 & \\
0 & & 0 & & 0 & & 0 \\
 & 0 & & \frac{(N+4)(N-1)}{2} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 0 & & &
\end{array} \tag{5.2}$$

which is in agreement with the number of fractional branes. The same result can be obtained by constructing toric resolutions. This suggests that the map between representations of the orbifold group and cycles of the resolution can be extended to this case, at least in the D-brane picture. It would be interesting to give a more precise description of this map, but we leave this for future work.

[20] In the computation of the orbifold cohomology performed in [31, section 2.2] one has to remove the $3(N-1)$ twisted sector $(1,1)$ classes corresponding to the constant forms of zero degree along the fixed lines. These give rise to non-normalizable forms on the resolved space.

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Finally, the case of orbifolds with discrete torsion is notably different. The exactly marginal operators correspond to complex structure deformations, as opposed to Kähler blow-up modes. Moreover, turning on these operators does not completely resolve the singularity [27, 46, 9, 31]. Therefore, the fractional branes cannot be given a clear geometric interpretation. We believe that these states can be mathematically described in an appropriate K-theoretic formalism, but we will not attempt to develop this here.

6. Fractional branes in the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold

The purpose of this section is to carry out the program outlined above for the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold. The moduli space of deformations can be thought as the Kähler moduli space of a non-compact Calabi-Yau threefold with a $\mathbb{P}^2$ cycle shrinking to zero size [21, 47]. As usual in $(2,2)$ superconformal models, the classical geometry is corrected by world-sheet instantons whose effects can be exactly summed using local mirror symmetry. Therefore, the first step in our analysis is to describe in detail the exact quantum moduli space. Then, using these exact results, we show how to continuously deform the fractional branes to the large radius limit.

6.1 The quantum moduli space

Since this is a non-conventional case — the threefold being non compact — we have in fact to use the local version of mirror symmetry developed in [21]–[24]. The coefficients of the marginal deformations can be thought as algebraic coordinates on the moduli space and the exact quantum geometry is described by periods of the local mirror geometry. The present model has been analyzed to various degrees in [21, 24], [47]–[50]. In the following we will review the construction of the local mirror model and present a detailed solution. The results in this subsection have been elaborated in collaboration with Albrecht Klemm to whom we are very grateful for valuable help.

The starting point is the linear sigma model construction of the (blown-up) orbifold background in IIA string theory. Following [51, 10], we consider a two dimensional $N = 2$ U(1) sigma model with four chiral fields X_i, $i = 0, \ldots, 3$ and charge vector

$$l = (-3, 1, 1, 1)\,. \tag{6.1}$$

The potential for the scalar fields is therefore

$$V = \left(|X_1|^2 + |X_2|^2 + |X_3|^2 - 3|X_0|^2 - r\right)^2, \tag{6.2}$$

where r is a Fayet-Iliopoulos parameter. The phase $r > 0$ corresponds to the blown-up phase, the exceptional $\mathbb{P}^2$ divisor being described by the coordinates X_1, X_2, X_3 which cannot vanish simultaneously. In the phase $r < 0$, X_0 cannot vanish and the

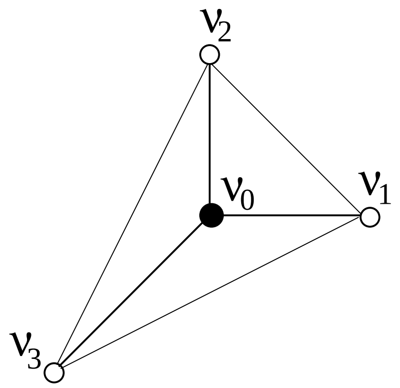

Figure 1: The trace of the non-complete fan corresponding to the blown-up $\mathbb{C}^3/\mathbb{Z}_3$. The black node in the center represents the compact exceptional divisor $D \simeq \mathbb{P}^2$.

exceptional divisor is blown down. It will be shown in the following that this picture is corrected quantum mechanically. In the blown-up phase, the moduli space of the linear sigma model is a non-compact toric variety described by the non-complete fan shown in figure 1.

The associated toric data are

$$\nu_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \nu_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \nu_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \nu_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}. \tag{6.3}$$

Local mirror symmetry associates to the non-compact toric variety represented in figure 1. a one dimensional local geometry described by the polynomial equation [23, 24]

$$\sum_{i=0}^{3} a_i y_i = 0\,, \tag{6.4}$$

where the variables y_i satisfy the constraint equation

$$y_1 y_2 y_3 y_0^{-3} = 1\,. \tag{6.5}$$

The solution can be easily presented in parametric form

$$y_1 = x_1^3\,, \qquad y_2 = x_2^3\,, \qquad y_3 = x_3^3\,, \qquad y_0 = x_1 x_2 x_3\,, \tag{6.6}$$

where x_1, x_2, x_3 are projective coordinates subject to the $\mathbb{C}^*$ identification $(x_1, x_2, x_3) \sim (\mu x_1, \mu x_2, \mu x_3)$. Note that there is also an $(\mathbb{C}^*)^3$ action on the space of parameters a_i

$$(a_0, a_1, a_2, a_3) \longrightarrow (\alpha_1\alpha_2\alpha_3 a_0, \alpha_1^3 a_1, \alpha_2^3 a_2, \alpha_3^3 a_3) \tag{6.7}$$

which leaves eq. (6.4) invariant. Therefore the moduli space is one dimensional and can be parameterized by the invariant coordinate

$$z = -27\frac{a_1 a_2 a_3}{a_0^3}\,, \tag{6.8}$$

where the factor -27 has been introduced for later convenience. The local mirror geometry is then described by the elliptic fibration

$$x_1^3 + x_2^3 + x_4^3 - \psi x_1 x_2 x_3 = 0\,, \tag{6.9}$$

with $z = 27/\psi^3$, which has been considered in a different context before [52, 53].

The exact solution of the model is provided by a three-dimensional vector of periods satisfying the differential equation[21]

$$\left[\theta_z^3 - z\left(\theta_z + \frac{1}{3}\right)\left(\theta_z + \frac{2}{3}\right)\theta_z\right] f = 0\,, \tag{6.10}$$

where $\theta_z = zd/dz$. This is a particular case of a general class of differential equations defining the Meijer G-functions $G^{m,n}_{p,q}\left(z \middle| \begin{smallmatrix} a_1 \cdots a_p \\ b_1 \cdots b_q \end{smallmatrix}\right)$ [54]

$$\left[(-1)^{p-m-n}\prod_{j=1}^{p}(\theta_z - a_j + 1) - \prod_{j=1}^{q}(\theta_z - b_j)\right] f = 0\,, \tag{6.11}$$

where $0 \leq n \leq p \leq q$ and $0 \leq m \leq q$. Note that after a change of variables $z \to -z$, eq. (6.10), becomes a Meijer equation with $p = q = 3$ and

$$a_1 = \frac{1}{3}\,, \qquad a_2 = \frac{2}{3}\,, \qquad a_3 = 1\,, \qquad b_1 = b_2 = b_3 = 0\,. \tag{6.12}$$

This shows that (6.10) has regular singular points at $z = 0, 1, \infty$. According to the general theory, first we have to find local solutions defined in a neighborhood of each singular point and then perform analytic continuation. The local solutions will be denoted by $f_i^x(z)$ where $x = 0, 1, \infty$ labels the singular points and $i = 0, 1, 2$ labels solutions in a given region. Note that eq. (6.10) admits a constant solution $f_0^x = 1$ which is defined everywhere. Before presenting the details of the other solutions, we would like to discuss some of their general properties.

The solutions to the indicial equations at the three singular points are $(0, 0, 0)$ at $z = 0$, $(0, 1, 1)$ at $z = 1$ and $(0, 1/3, 2/3)$ at $z = \infty$. Accordingly, we expect the solutions near $z = 0$ to be logarithmic. At $z = 1$ we expect a power series solution with index 1 and a logarithmic solution, while at $z = \infty$ both solutions are expected to be power series involving fractional powers. Therefore we will obtain non-trivial monodromy transformations about these points. Moreover, the fact that all solutions at $z = \infty$ are power series signals an exactly solvable conformal field theory associated to that point. This is the perturbative orbifold point, similar to the Gepner points in the one parameter models considered in [55, 56, 57].

[21] Note that this equation is derived by applying the general mirror construction to this local case. This is not the Picard-Fuchs equation for the periods of the elliptic curve (6.9) which will enter the discussion later.

In the context of global mirror symmetry, it is known that one can define a special basis of solutions such that the monodromy transformations are integral symplectic matrices. Using special coordinates t^a on the moduli space, this basis can be written in general as $(1, t^a, \frac{\partial F}{\partial t^a}, 2F - t^a \frac{\partial F}{\partial t^a})$ where $F(t^a)$ is the $N = 2$ prepotential. The functions $(t^a, \frac{\partial F}{\partial t^a})$ represent the periods of the holomorphic three-form of the mirror manifold with respect to a symplectic basis of cycles.

In the local case, we have only three periods, therefore we cannot find a symplectic basis. This is consistent with the local mirror geometry (6.9) being one dimensional. In fact, as observed in [47, 50], eq. (6.10) is the logarithmic integral of an ordinary hypergeometric equation

$$\left[\theta_z^2 - z\left(\theta_z + \frac{1}{3}\right)\left(\theta_z + \frac{2}{3}\right)\right] f = 0\,. \tag{6.13}$$

This represents the Picard-Fuchs equation for the periods of the global holomorphic one-form on a symplectic basis of cycles of the elliptic curve (6.9). As before, we have regular singular points at $(0, 1, \infty)$, the monodromy group being isomorphic to $\Gamma(3)$ [58]. Therefore we can define a special basis of solutions of the form $(1, t(z), t_d(z))$ [50] such that the monodromy transformations are integral 3×3 matrices of the form

$$\begin{bmatrix} 1 & 0 \\ m & M \end{bmatrix}. \tag{6.14}$$

Here $M \in \Gamma(3)$ represents the $\Gamma(3)$ monodromy acting on (t, t_d) which are logarithmic integrals of the solutions of (6.13). The integral column vector m reflects the fact that the periods (t, t_d) may pick up constant shifts when analytically continued about the singular points. This will play an important role at a latter stage. We now derive explicit formulae for the local solutions and determine the monodromy.

(i) Solutions at $z = 0$.

The other two solutions in the region $|z| < 1$ have been given in integral form in [54]. We have one logarithmic solution $f_1^0(z) = G_{3,3}^{2,2}\left(-z \left| \begin{smallmatrix} 1/3 & 2/3 & 1 \\ 0 & 0 & 0 \end{smallmatrix} \right.\right)$ which can be represented in integral form[22]

$$\begin{aligned} G_{3,3}^{2,2}\left(-z \left| \begin{matrix} 1/3 & 2/3 & 1 \\ 0 & 0 & 0 \end{matrix} \right.\right) &= \frac{1}{2\pi i \Gamma(1/3)\Gamma(2/3)} \times \\ &\times \int_C \frac{\Gamma(-s)\Gamma(s+1/3)\Gamma(s+2/3)}{s\Gamma(1+s)} (-z)^s ds\,, \end{aligned} \tag{6.15}$$

where $|z| < 1$ and $|\arg(z)| < \pi$. The contour C is parallel with the imaginary axis, encircles the origin and closes to the right as shown in the figure 2.

[22] This function differs by a normalization factor $-\frac{1}{\Gamma(1/3)\Gamma(2/3)}$ from the original G-function of [54]. We have included this factor for latter convenience.

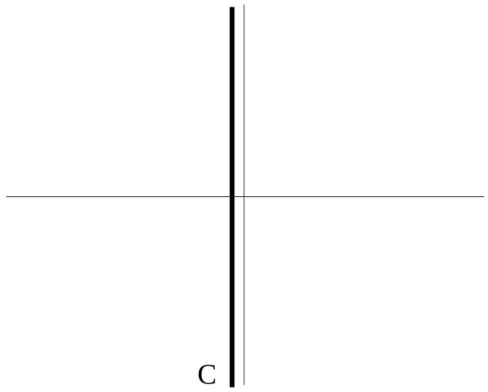

Figure 2: The integration contour for Meijer's G-functions.

By evaluating the residues, we can rewrite (6.15) as a power series

$$f_1^0(z) = \log\left(-\frac{z}{27}\right) + \frac{1}{\Gamma(1/3)\Gamma(2/3)} \sum_{n=1}^{\infty} \frac{\Gamma(n+1/3)\Gamma(n+2/3)}{n\Gamma(1+n)^2} z^n \,. \tag{6.16}$$

On general grounds, we expect one extra independent solution in order to obtain a complete system. It turns out that there exist two particular double logarithmic solutions[23]

$$\begin{aligned} G_{3,3}^{3,1}\left(-z \left| \begin{matrix} 1/3 & 2/3 & 1 \\ 0 & 0 & 0 \end{matrix} \right.\right) &= \frac{1}{2\pi i}\int_C \frac{\Gamma(-s)^2\Gamma(2/3+s)}{s\Gamma(2/3-s)}(-z)^s ds\,, \\ G_{3,3}^{3,1}\left(-z \left| \begin{matrix} 2/3 & 1/3 & 1 \\ 0 & 0 & 0 \end{matrix} \right.\right) &= \frac{1}{2\pi i}\int_C \frac{\Gamma(-s)^2\Gamma(1/3+s)}{s\Gamma(1/3-s)}(-z)^s ds\,, \end{aligned} \tag{6.17}$$

where the integration is taken along the same contour C as above and $|z| < 1$ and $|\arg(z)| < \pi$. Adopting the notation of [54], these functions will be denoted by $G_{3,3}^{3,1}(-z\,||1/3\,)$ and respectively $G_{3,3}^{3,1}(-z\,||2/3\,)$. Again, we can evaluate the residues and obtain the following series expansions:

$$\begin{aligned} G_{3,3}^{3,1}\left(-z\left|\left|\frac{1}{3}\right.\right.\right) &= \frac{1}{2}\log^2\left(-\frac{z}{27}\right) + \frac{\pi\sqrt{3}}{3}\log\left(-\frac{z}{27}\right) + \frac{\pi^2}{3} + \\ &+ \frac{1}{\Gamma(1/3)\Gamma(2/3)}\sum_{n=1}^{\infty}\frac{\Gamma(n+1/3)\Gamma(n+2/3)}{n\Gamma(1+n)^2} z^n \log\left(-\frac{z}{27}\right) + \\ &+ \frac{1}{\Gamma(1/3)\Gamma(2/3)}\sum_{n=1}^{\infty}\left[n\left(\Psi\left(\frac{2}{3}+n\right) + \Psi\left(\frac{2}{3}-n\right) - \right.\right. \\ &\qquad \left.\left. -2\Psi(1+n) + 3\log 3\right) - 1\right]\frac{\Gamma(1/3+n)\Gamma(2/3+n)}{n^2\Gamma(1+n)^2} z^n\,. \end{aligned} \tag{6.18}$$

$$G_{3,3}^{3,1}\left(-z\left|\left|\frac{2}{3}\right.\right.\right) = \frac{1}{2}\log^2\left(-\frac{z}{27}\right) - \frac{\pi\sqrt{3}}{3}\log\left(-\frac{z}{27}\right) + \frac{\pi^2}{3} +$$

[23] These solutions differ by a $-$ sign from the original G-functions of [54].

$$+\frac{1}{\Gamma(1/3)\Gamma(2/3)}\sum_{n=1}^{\infty}\frac{\Gamma(n+1/3)\Gamma(n+2/3)}{n\Gamma(1+n)^2}z^n\log\left(-\frac{z}{27}\right)+$$

$$+\frac{1}{\Gamma(1/3)\Gamma(2/3)}\sum_{n=1}^{\infty}\left[n\left(\Psi\left(\frac{1}{3}+n\right)+\Psi\left(\frac{1}{3}-n\right)-\right.\right. \tag{6.19}$$

$$\left.\left.-2\Psi(1+n)+3\log 3\right)-1\right]\frac{\Gamma(1/3+n)\Gamma(2/3+n)}{n^2\Gamma(1+n)^2}z^n\,.$$

Note that the three solutions presented in (6.16) and (6.18) are not independent. They satisfy the linear dependence relation

$$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)G^{2,2}_{3,3}\left(-z\left|\begin{array}{ccc}1/3 & 2/3 & 1\\ 0 & 0 & 0\end{array}\right.\right)=$$

$$=G^{3,1}_{3,3}\left(-z\left|\left|\frac{1}{3}\right.\right.\right)-G^{3,1}_{3,3}\left(-z\left|\left|\frac{2}{3}\right.\right.\right). \tag{6.20}$$

A general local solution near $z=0$ can therefore be written as any linear combination of these three functions subject to the constraint (6.20). For latter convenience, we will consider the following linear combination

$$f^0_2(z)=\frac{1}{2}\left(G^{3,1}_{3,3}\left(-z\left|\left|\frac{1}{3}\right.\right.\right)+G^{3,1}_{3,3}\left(-z\left|\left|\frac{2}{3}\right.\right.\right)\right). \tag{6.21}$$

This can be rewritten in the form

$$f^0_2(z)=\frac{1}{2}f^0_1(z)^2-\frac{1}{12}+\mathrm{O}(z)\,. \tag{6.22}$$

Note that the above solutions are absolutely convergent in the region $|z|<1$. If $|z|=1$ the series are convergent for $z\neq 1$. However, the simple logarithmic solution (6.15) diverges at $z=1$, which is a singular point similar to the conifold point in the quintic moduli space [55]. This singular point will play an important role in determining the integral basis of solutions $(1,t,t_d)$. According to [50], the coordinate t can be identified with the simple logarithmic solution (6.16). More precisely, we will take

$$t(z)=\frac{1}{2\pi i}f^0_1(z)\,. \tag{6.23}$$

The dual period $t_d=\partial F/\partial t$ can be found by studying the system of solutions near the singular point $z=1$ and analytically continuing the vanishing period [55].

(ii) Solutions at $z=1$.

A basis of solutions near this singular point can be found by changing variables $u=1-z$ in (6.10). This results in the following differential equation:

$$\left(\theta_u^3-\frac{2+u}{1-u}\theta_u^2+\frac{1}{9}\frac{2u^2-2u+9}{(1-u)^2}\theta_u\right)f=0 \tag{6.24}$$

which can be solved with standard recursive methods (see for example [58]). As mentioned before, the solutions of the indicial equation are $(0,1,1)$, therefore one of the solutions should be a power series of index one, while the second solution should be logarithmic. It can be shown [50] that the logarithmic solution is given by the analytic continuation of the period $t(z)$ to $z=1$

$$t(u) = -3t_d(u)\log(u) + \mathrm{O}(1)\,. \tag{6.25}$$

The function

$$t_d(u) = -\frac{\sqrt{3}}{6}\left(u + \frac{11}{18}u^2 + \cdots\right) \tag{6.26}$$

is itself a solution of the equation and represents the vanishing period at the singular point. It is related by analytic continuation to the dual period $t_d(z)$. Using these facts, one can find the precise form of the period $t_d(z)$ near $z=0$

$$t_d(z) = -\frac{1}{4\pi^2}f_2^0(z) - \frac{1}{2}t(z) + \frac{1}{3} = \frac{1}{2}t(z)^2 - \frac{1}{2}t(z) + \frac{1}{4} + \mathrm{O}(z)\,, \tag{6.27}$$

where $f_2^0(z)$ has been defined in (6.21).

(iii) Solutions at $z=\infty$.

The local solutions in the region $|z|>1$ can be found similarly, by changing variables $\zeta = 1/z$ and then solving the resulting equation

$$\left[\zeta{\theta_\zeta}^3 - \left(\theta_\zeta - \frac{1}{3}\right)\left(\theta_\zeta - \frac{2}{3}\right)\theta_\zeta\right] f = 0\,. \tag{6.28}$$

This is again a Meijer equation whose local solutions can be determined by analytically continuing the functions $G^{3,1}_{3,3}\left(-z\,||1/3\,\right)$ and $G^{3,1}_{3,3}\left(-z\,||2/3\,\right)$ in a neighborhood of $z=\infty$ [54]. We obtain $f_1^\infty(\zeta) = E_{3,3}\left(-1/\zeta\,||2/3\,\right)$, $f_2^\infty(\zeta) = E_{3,3}\left(-1/\zeta\,||1/3\,\right)$, where

$$\begin{aligned}
E_{3,3}\left(-\frac{1}{\zeta}\,\middle|\middle|\,\frac{2}{3}\right) &= -3\frac{\Gamma\left(1/3\right)^2}{\Gamma\left(2/3\right)}e^{\pi i/3}\zeta^{1/3}{}_3F_2\left(\frac{1}{3},\frac{1}{3},\frac{1}{3};\frac{2}{3},\frac{4}{3};\zeta\right),\\
E_{3,3}\left(-\frac{1}{\zeta}\,\middle|\middle|\,\frac{1}{3}\right) &= -\frac{9}{2}\frac{\Gamma\left(2/3\right)^2}{\Gamma\left(1/3\right)}e^{2\pi i/3}\zeta^{2/3}{}_3F_2\left(\frac{2}{3},\frac{2}{3},\frac{2}{3};\frac{4}{3},\frac{5}{3};\zeta\right).
\end{aligned} \tag{6.29}$$

These solutions are well defined and convergent if $|\zeta|<1$ and $|\arg(\zeta)|<\pi$.

It follows that the analytic continuation of the periods (t, t_d) in a neighborhood of ∞ reads

$$\begin{aligned}
t(\zeta) &= \frac{\omega^2-\omega}{4\pi^2}\left(E_{3,3}\left(-\frac{1}{\zeta}\,\middle|\middle|\,\frac{2}{3}\right) - E_{3,3}\left(-\frac{1}{\zeta}\,\middle|\middle|\,\frac{1}{3}\right)\right),\\
t_d(\zeta) &= \frac{1}{3} + \frac{1}{4\pi^2}\left(\omega E_{3,3}\left(-\frac{1}{\zeta}\,\middle|\middle|\,\frac{2}{3}\right) + \omega^2 E_{3,3}\left(-\frac{1}{\zeta}\,\middle|\middle|\,\frac{1}{3}\right)\right),
\end{aligned} \tag{6.30}$$

where $\omega = e^{2\pi i/3}$. This can be proved by direct computation using (6.23), (6.27), and (6.20) and taking into account the fact that $\frac{1}{2\pi i\Gamma(1/3)\Gamma(2/3)} = \frac{\omega^2-\omega}{4\pi^2}$. Note that the values of the periods at the singular point are

$$t\,(\zeta = 0) = 0\,, \qquad t_d\,(\zeta = 0) = \frac{1}{3}\,. \tag{6.31}$$

Once we know the local solutions near each singular point, the global solution can be obtained by patching them together. Since the periods are multivalued functions, this process requires branch cuts in the complex plane. These are implicit in the restrictions imposed on the phase of z in the paragraphs following (6.15) and (6.29) which represent two branch cuts joining $z = 0$ and $z = 1$ and respectively $z = 1$ and $z = \infty$ along the real axis. This will allow us to assign unambiguous values of the periods to all points in the moduli space (away from the cuts).

(iv) Monodromy.

The monodromy of the integral basis of solutions around the singular points can be determined from the explicit local solutions derived above. By performing a counterclockwise rotation about each of the three singular points, we find

$$M_0 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \qquad M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix},$$
$$M_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -2 & -3 \\ 0 & 1 & 1 \end{bmatrix} \tag{6.32}$$

satisfying $M_\infty = M_1 M_0$. Note that M_0 and M_1 are of infinite order and they agree with the expected behavior of the periods in the infinite volume limit and respectively at the conifold point. However, M_∞ is of third order, reflecting the quantum $\mathbb{Z}_3$ symmetry of the orbifold point as explained in [21]. This identifies the orbifold point as a quotient singularity in the moduli space.

6.2 Fractional branes and BPS states on the moduli space

We now show how to use the above detailed solution in order to relate the orbifold fractional branes to the generic BPS states on the moduli space. This will allow us to deform this states along a path between the orbifold point and the large radius limit and to finally interpret them as macroscopic branes on supersymmetric cycles.

In order to carry out this program, we need to know the form of the complexified Kähler class $t_b \equiv B + iJ$ in terms of the exact periods $(1, t, t_d)$. This is essentially the problem of finding a precise form of the mirror map. According to the properties of the monomial-divisor map as discussed for example in [47], the asymptotic form of t_b is

$$t_b \sim \frac{1}{2\pi i}\log\left(\pm\frac{z}{27}\right) + \mathrm{O}(z)\,. \tag{6.33}$$

Since the period t has an asymptotic behavior $\sim \log(-z/27)$ this fixes $t_b = t + C$, where C is an integration constant which depends on the choice of sign of $z/27$ inside the logarithm. It has been argued in [47] that this sign should be positive and C was fixed to $-1/2$

$$t_b = t - \frac{1}{2}\,. \tag{6.34}$$

With this choice for the mirror map, the quantum volume of the $\mathbb{P}^1$ cycle in the exceptional divisor is positive everywhere on the moduli space. In particular the value of t_b at the orbifold point is $1/2$.[24]

Note that, in the basis $(1, t_b, t_d)$, the monodromy matrices will no longer be integral since t_b and t_d are shifted by half integral numbers. Therefore, the BPS states will be represented with respect to the basis $(1, t_b, t_d)$ by charge vectors (n_0, n_1, n_2) where n_1, n_2 are integral and n_0 is half integral. The associated central charge is

$$Z(n_0, n_1, n_2) = n_0 + n_1 t_b + n_2 t_d\,. \tag{6.35}$$

Our first goal is to identify the charge vectors associated with the three boundary states constructed in the previous section. Then, using analytic continuation to the large radius limit, we will interpret the resulting BPS states as D-branes wrapped on cycles.

The first step is to note that the perturbative $\mathbb{C}^3/\mathbb{Z}_3$ orbifold CFT has a $\mathbb{Z}_3$ quantum symmetry [21] which permutes the twist fields cyclically. Taking into account the boundary state construction in the previous sections, the quantum $\mathbb{Z}_3$ also permutes the fractional branes in a similar manner. Therefore all boundary states, associated with the three irreducible representations of $\mathbb{Z}_3$ γ_1, γ_2 and γ_3, form an orbit of the discrete global symmetry group. Changing perspective, the same quantum symmetry manifests itself in the form of a third order monodromy of periods on the moduli space as explained in [21] and in the previous paragraph. From here we conclude that the fractional branes are naturally in one to one correspondence with a set of three periods forming an orbit of the monodromy generator M_∞.

The next available piece of information is that all fractional branes have equal mass, which is $1/3$ of the mass of a D0-brane. At the same time the formula (6.31) shows that this is precisely the mass of a state with charges $(0, 0, \pm 1)$ at the orbifold. Therefore we will identify the three fractional branes with the following states:[25]

$$(0,0,1)\,, \qquad \left(\frac{1}{2}, 1, 1\right), \qquad \left(\frac{1}{2}, -1, -2\right) \tag{6.36}$$

[24] Note that fixing the integration constant is physically meaningful. We will show later that the present choice corresponds to turning on a half integral B-field on the hyperplane cycle of $\mathbb{P}^2$.

[25] The above arguments do not fix the sign of the charges. We have chosen the sign so that the three states will have a total D0-brane charge 1 rather than -1, as it will be clear latter. A different choice of sign would correspond to antiparticle states.

obtained by acting[26] by the monodromy generator M_∞. The corresponding central charges read

$$Z(0,0,1) = t_d\,, \qquad Z\left(\frac{1}{2},1,1\right) = \frac{1}{2} + t_b + t_d\,,$$
$$Z\left(\frac{1}{2},-1,-2\right) = \frac{1}{2} - t_b - 2t_d\,. \tag{6.37}$$

Clearly, all states have mass equal to 1/3 of the D0-brane mass which is normalized to one.

In order to complete the analysis, we have to understand if the proposed BPS states have a well defined D-brane interpretation in the large radius limit. Although the periods can be continued along any given path joining the two points, the continuation of the BPS states is more subtle due to the possible jumping phenomena. This phenomenon and the associated marginal stability curves have been studied intensively in the context of Seiberg-Witten solutions [59, 60, 61]. In the present case, note that all periods have real values at the orbifold point, therefore all marginal stability curves will necessarily pass through that point. Therefore, as long as the curves are reasonably shaped (that is if they have no self intersection), it is possible to find a path between the orbifold point and the large radius limit that avoids them. In the following we will assume that this is in fact the case and that there exists such a path along which the states (6.36) are stable. The final result will be shown to be consistent with this assumption.

The asymptotic expansion of the central charges of the three BPS states reads

$$Z(0,0,1) = \frac{1}{2}t_b^2 + \frac{1}{8} + \mathrm{O}(z)\,, \qquad Z\left(\frac{1}{2},1,1\right) = \frac{1}{2}t_b^2 + t_b + \frac{5}{8} + \mathrm{O}(z)\,,$$
$$Z\left(\frac{1}{2},-1,-2\right) = -t_b^2 - t_b + \frac{1}{4} + \mathrm{O}(z)\,. \tag{6.38}$$

The corresponding D-brane states can be identified by comparing (6.38) to the semiclassical expression for the central charge of a state with effective D-brane charges (q_0, q_2, q_4)

$$Z(q_0,q_2,q_4) = -q_4\frac{t_b^2}{2} + q_2 t_b + q_0\,. \tag{6.39}$$

Note that, according to the discussion in section 4, the D-brane states are classified by the K-theory group $K(D)$, where $D \simeq \mathbb{P}^2$ is the exceptional divisor. The vector $Q = (q_0, q_2, q_4)$ takes values in the total cohomology space $H^4(D,\mathbf{Q}) \oplus H^2(D,\mathbf{Q}) \oplus H^0(D,\mathbf{Q})$ and it measures the effective charges of a brane configuration represented by a given K-theory class. We consider rational cohomology since the effective charges may be fractional. Given a K-theory class represented by a vector bundle (or, more

[26] Note that the monodromy matrices act on the charge vectors by right multiplication and on the period vectors by left multiplication.

generally, a coherent sheaf) V on D, we can determine Q from the Chern-Simons couplings found in [62, 63, 64]. A careful analysis of these couplings, taking into account the twisting of the worldvolume fermions in normal directions, has been performed in [65]. [38, 65] According to the results therein, the vector of induced charges for a system of Dp-branes wrapping a supersymmetric cycle D is given by

$$Q = \mathrm{ch}(V)\sqrt{\frac{\hat{A}(T_D)}{\hat{A}(N_D)}}\,. \tag{6.40}$$

Here T_D and N_D denote the tangent and respectively the normal bundle to the cycle D[27]

For the case when D is a holomorphic surface embedded in Calabi-Yau threefold, we have [65]

$$\sqrt{\frac{\hat{A}(T_D)}{\hat{A}(N_D)}} = 1 + \frac{1}{48}\left(p_1(N_D) - p_1(T_D)\right) = 1 + \frac{\chi(D)}{24}w_D\,, \tag{6.41}$$

where $\chi(D)$ is the topological Euler characteristic of D and w_D is the fundamental class. Therefore we obtain

$$Q = r + c_1(V) + \left(\frac{r}{8} + \frac{1}{2}c_1^2(V) - c_2(V)\right) w_D\,. \tag{6.42}$$

Using this formula, we can show that the D-brane configurations corresponding to the three central charges (6.38) are

$$\begin{aligned} Z(0,0,1) &\longrightarrow \overline{\mathrm{D4}}\,, \qquad Z\left(\frac{1}{2},1,1\right) \longrightarrow \overline{\mathrm{D4}} + \mathrm{D2}\,, \\ Z\left(\frac{1}{2},-1,-2\right) &\longrightarrow 2\mathrm{D4} + \overline{\mathrm{D2}} + \mathrm{D0}\,. \end{aligned} \tag{6.43}$$

In the above, the symbol D4 represents a D4-brane wrapped on the exceptional divisor D while D2 represents a D2-brane wrapped on a $\mathbb{P}^1 \subset \mathbb{P}^2$ cycle in the hyperplane class H. Barred symbols denote antibrane states which correspond to "negative" K-theory classes. These states correspond to the fractional branes at the orbifold.

[27]There is a subtlety related to this formula which has been clarified in [66]. Since $\mathbb{P}^2$ is not a spin manifold, but only a spin^c-manifold, the bundle V on D is actually a spin^c-bundle. This means that the curvature $\mathrm{Tr}(F)$ is a half-integral class rather than integral, as in the conventional case. However, in the present case, we claim that there is a flat background B-field turned on such that $\int_{\mathbb{P}^1} B \in \mathbb{Z} + 1/2$. The presence of this B-field is related to the choice of the integration constant of the mirror map discussed after equation (6.33). The net effect is to cancel the effect of $w_2(D)$ discussed in [66], so that V can be regarded as a conventional vector bundle in (6.40).

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The first two configurations are represented by the classes $-[\mathcal{O}(-1)]$ and $-[\mathcal{O}]$ where $\mathcal{O}$, $\mathcal{O}(-1)$ are the trivial and respectively the tautological line bundle on $\mathbb{P}^2$. The third case deserves more attention since it corresponds to a rank 2 holomorphic bundle V on $\mathbb{P}^2$ characterized by

$$r = 2\,, \qquad c_1(V) = -1\,, \qquad c_2(V) = 1\,. \tag{6.44}$$

As a consistency check we should now be able to check if the resulting D-brane states are indeed BPS. According to [64], this means that the three bundles should be holomorphic and stable. These criteria are clearly satisfied by the first two line bundles. However, this is a non-trivial test for the rank 2 bundle V [67, 68]. According to the classification therein, it can be checked that the bundle V in our problem is an exceptional stable holomorphic bundle on $\mathbb{P}^2$. This means that it is holomorphic and stable and it has no deformations, i.e. the moduli space reduces to a single point. It is remarkable that our BPS states analysis has lead precisely to one of these exceptional bundles which form a special discrete series. Moreover, note that both the fractional branes and the D-brane configurations (6.43) have no moduli. This is in agreement with the arguments of [25] which show that the number of moduli of the BPS states should be preserved under Kähler deformations.

To this end, note that there is one more test we can perform. In [31], Douglas and Fiol have introduced the index

$$\mathrm{tr}_{\mathcal{H}_{\mathrm{open}}}\left((-1)^F e^{-2tH_o}\right) \tag{6.45}$$

that counts the number of fermion zero modes in the Ramond open string sector in the presence of D-brane states. They have also argued on the basis of Dirac quantization condition, that this index should actually compute the classical intersection number of supersymmetric cycles up to sign. More precisely, the index in (6.45) can be given a closed string interpretation using the formalism of boundary states discussed previously. In fact (6.45) is related by a modular transformation to the following closed string amplitude

$$_{\mathrm{RR}}\langle B_1|e^{-lH_c}|B_2\rangle_{\mathrm{RR}}\,, \tag{6.46}$$

where $|B_{1,2}\rangle$ are two arbitrary boundary states. This formula defines an antisymmetric bilinear form on the set of all boundary states which is the equivalent of an intersection form. In the case of even branes, this seems to lead to a puzzle since the classical intersection form is symmetric. The resolution resides in the fact that the open string index computes the intersection form of the branes as seen from the dual mirror point of view [25]. The exact periods (t_b, t_d) found in the previous subsection, are really sections of a rank two holomorphic vector bundle E on the moduli space with structure group $\Gamma(3) \subset \mathrm{SL}(2,\mathbb{Z})$. The charge (n_1, n_2) vectors of BPS states are locally constant sections and we can define a natural symplectic form

$$\omega : ((n_1, n_2), (n_1', n_2')) \longrightarrow 3(n_1' n_2 - n_1 n_2')\,. \tag{6.47}$$

This represents the intersection form on the middle homology of the elliptic curve (6.9). The coefficient 3 has been chosen to agree with the intersection number $H \cdot D = -3$.

In our situation the index (6.45) can be easily computed at the orbifold point using conformal field theory techniques. As noted in [31], it turns out that for a given pair of fractional branes one actually counts the net number of chiral fermion multiplets in the RR sector of the open string stretching between the branes which are left invariant by the orbifold projection. Therefore, in the presence of two fractional branes classified by two irreducible representations γ_a, γ_b, we count the number of chiral fermion multiplets χ satisfying

$$\gamma(g)_a^{-1}\chi\gamma(g)_b = R(g)\chi\,, \tag{6.48}$$

where g is a generator of $\mathbb{Z}_3$ and R is the standard three dimensional representation defined by embedding in SU (3). The result has a concise graphical description encoded in the quiver diagram corresponding to the regular representation.

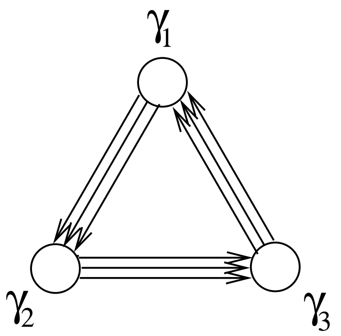

Figure 3: The quiver diagram for the regular representation of $\mathbb{C}^3/\mathbb{Z}_3$.

Given two irreducible representations γ_a, γ_b the (a, b)-th entry of the resulting intersection form is given by the number of edges connecting the two vertices. The sign is given by the orientation of the edges: the contribution is positive if the arrows point from the vertex a to the vertex b and negative in the reversed situation. This gives the following antisymmetric intersection form:

	γ_1	γ_2	γ_3
γ_1	0	3	-3
γ_2	-3	0	3
γ_3	3	-3	0

An elementary computation shows that this agrees with the intersection form (6.47) evaluated on the charges (6.36).

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A. Conformal field theory conventions

In this appendix we summarize some of the conventions we have used as well as properties of ϑ functions that are needed for the construction of the boundary states. For the flat space discussion we have used the following mode expansions for the open-string fields

$$
\begin{aligned}
X^\mu &= x^\mu + 2\pi p^\mu t + i\sqrt{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-i\pi n t}\cos(n\pi\sigma)\,, \qquad \mu = 0,\dots,p\,,\\
\psi^M &= \sqrt{\pi}\sum_r \psi_r^M e^{-i\pi r(t-\sigma)}\,, \qquad M = 0,\dots,9\,,\\
\widetilde{\psi}^\mu &= \sqrt{\pi}\sum_r \psi_r^\mu e^{-i\pi r(t+\sigma)}\,, \quad X^i = i\sqrt{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^i e^{-i\pi n t}\sin(n\pi\sigma)\,, \quad i = p+1,\dots,9\,,\\
\widetilde{\psi}^i &= -\sqrt{\pi}\sum_r \psi_r^i e^{-i\pi r(t+\sigma)}\,,
\end{aligned}
\tag{A.1}
$$

where as usual $r \in \mathbb{Z}$ in the R sector and $r \in \mathbb{Z}+1/2$ in the NS sector and $0 \leq \sigma \leq 1$. For the closed string the expansions are

$$
\begin{aligned}
X^M &= x^M + 2\pi p^\mu t + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\left(\frac{1}{n}\alpha_n^M e^{-2\pi i n(t-\sigma)} + \widetilde{\alpha}_n^M e^{-2\pi i n(t+\sigma)}\right), \quad M = 0,\dots,9\,,\\
\psi^M &= \sqrt{2\pi}\sum_r \psi_r^M e^{-2\pi i r(t-\sigma)}\,, \qquad \widetilde{\psi}^M = \sqrt{2\pi}\sum_r \psi_r^M e^{-2\pi i r(t+\sigma)}\,,
\end{aligned}
\tag{A.2}
$$

and the oscillator algebra is as usual

$$
\begin{aligned}
\left[\alpha_n^M, \alpha_m^N\right] &= \left[\widetilde{\alpha}_n^M, \widetilde{\alpha}_m^N\right] = n\delta_{n+m}\delta_{MN}\,,\\
\left\{\psi_r^M, \psi_s^N\right\} &= \{\widetilde{\psi}_r^M, \widetilde{\psi}_s^N\} = \delta_{r+s}\delta_{MN}\,.
\end{aligned}
\tag{A.3}
$$

The action of the orbifold group Γ is natural on the complex coordinates

$$
Z^i = \frac{1}{\sqrt{2}}(X^{2i} + iX^{2i+1})\,, \qquad i = 1,2,3\,,
\tag{A.4}
$$

$$
\bar{Z}^i = \frac{1}{\sqrt{2}}(X^{2i} - iX^{2i+1})\,, \qquad \lambda^i = \frac{1}{\sqrt{2}}(\psi^{2i} + i\psi^{2i+1})\,, \qquad \bar{\lambda}^i = \frac{1}{\sqrt{2}}(\psi^{2i} - i\psi^{2i+1})\,,
$$

so that the oscillator expansion of these fields in the the closed string is

$$
\begin{aligned}
Z^i &= z^i + 2\pi p^i t + \frac{i}{\sqrt{2}} \sum_{n\neq 0} \left(\frac{1}{n} \beta^i_n e^{-2\pi i n(t-\sigma)} + \widetilde{\beta}^i_n e^{-2\pi i n(t+\sigma)} \right), \\
\bar{Z}^i &= \bar{z}^i + 2\pi \bar{p}^i t + \frac{i}{\sqrt{2}} \sum_{n\neq 0} \left(\frac{1}{n} \bar{\beta}^i_n e^{-2\pi i n(t-\sigma)} + \widetilde{\bar{\beta}}^i_n e^{-2\pi i n(t+\sigma)} \right), \\
\lambda^i &= \sqrt{2\pi} \sum_r \lambda^i_r e^{-2\pi i r(t-\sigma)}, \qquad \bar{\lambda}^i = \sqrt{2\pi} \sum_r \widetilde{\lambda}^i_r e^{-2\pi i r(t+\sigma)}, \\
\widetilde{\lambda}^i &= \sqrt{2\pi} \sum_r \widetilde{\lambda}^i_r e^{-2\pi i r(t+\sigma)}, \qquad \widetilde{\bar{\lambda}}^i = \sqrt{2\pi} \sum_r \widetilde{\bar{\lambda}}^i_r e^{-2\pi i r(t+\sigma)},
\end{aligned} \tag{A.5}
$$

with commutation relations

$$
[\beta^i_n, \bar{\beta}^j_m] = [\widetilde{\beta}^i_n, \widetilde{\bar{\alpha}}^j_m] = n\delta_{n+m}\delta_{ij}, \qquad \{\lambda^i_r, \bar{\lambda}^j_s\} = \{\widetilde{\lambda}^i_r, \widetilde{\bar{\lambda}}^j_s\} = \delta_{r+s}\delta_{ij} \tag{A.6}
$$

with the rest of (anti)commutators vanishing.

When closed strings are in an orbifold background, their oscillator modding change, since the string is identified when going once around σ via the orbifold group. For the $\mathbb{C}^3/\mathbb{Z}_N$ orbifold with action

$$
\begin{aligned}
Z^i &\longrightarrow e^{2\pi i \nu_i} Z^i, \qquad \bar{Z}^i \longrightarrow e^{-2\pi i \nu_i} \bar{Z}^i, \\
\lambda^i &\longrightarrow e^{2\pi i \nu_i} \lambda^i, \qquad \bar{\lambda}^i \longrightarrow e^{-2\pi i \nu_i} \bar{\lambda}^i,
\end{aligned} \tag{A.7}
$$

with $\nu_1 + \nu_2 + \nu_3 = 0(1)$, the m-twisted sector the worldsheet fields have the following expansion:

$$
\begin{aligned}
Z^i &= \frac{i}{\sqrt{2}} \sum_{n\neq 0} \left(\frac{1}{n+m\nu_i} \beta^i_{n+m\nu_i} e^{-2\pi i(n+m\nu_i)(t-\sigma)} + \frac{1}{n-m\nu_i} \widetilde{\beta}^i_{n-m\nu_i} e^{-2\pi i(n-m\nu_i)(t+\sigma)} \right), \\
\bar{Z}^i &= \frac{i}{\sqrt{2}} \sum_{n\neq 0} \left(\frac{1}{n-m\nu_i} \bar{\beta}_{n-m\nu_i} e^{-2\pi i(n-m\nu_i)(t-\sigma)} + \frac{1}{n+m\nu_i} \widetilde{\bar{\beta}}_{n+m\nu_i} e^{-2\pi i(n+m\nu_i)(t+\sigma)} \right), \\
\lambda^i &= \sqrt{2\pi} \sum_r \lambda^i_{r-m\nu_i} e^{-2\pi i(r-m\nu_i)(t-\sigma)}, \qquad \bar{\lambda}^i = \sqrt{2\pi} \sum_r \widetilde{\lambda}^i_{r-m\nu_i} e^{-2\pi i(r-m\nu_i)(t+\sigma)}, \\
\widetilde{\lambda}^i &= \sqrt{2\pi} \sum_r \widetilde{\lambda}^i_{r-m\nu_i} e^{-2\pi i(r-m\nu_i)(t+\sigma)}, \\
\widetilde{\bar{\lambda}}^i &= \sqrt{2\pi} \sum_r \widetilde{\bar{\lambda}}^i_{r+m\nu_i} e^{-2\pi i(r+m\nu_i)(t+\sigma)},
\end{aligned} \tag{A.8}
$$

with the following commutation relations:

$$
\begin{aligned}
[\beta^i_{n+m\nu_i}, \bar{\beta}^j_{m-m\nu_i}] &= [\widetilde{\beta}^i_{n-m\nu_i}, \widetilde{\bar{\alpha}}^j_{m+m\nu_i}] = (n+m)\nu_i \delta_{n+m}\delta_{ij}, \\
\{\lambda^i_{r+m\nu_i}, \bar{\lambda}^j_{s-m\nu_i}\} &= \{\widetilde{\lambda}^i_{r-m\nu_i}, \widetilde{\bar{\lambda}}^j_{s+m\nu_i}\} = \delta_{r+s}\delta_{ij}.
\end{aligned} \tag{A.9}
$$

The cylinder amplitude computation in the boundary state formalism can be performed by using the explicit expressions for the boundary states we have found and the closed string hamiltonian in the m-th twisted sector

$$
\begin{aligned}
H_c = \pi p^2 + 2\pi \sum_{\mu=0,1} \left(\sum_{n=1}^{\infty} \alpha^{\mu}_{-n} \alpha^{\mu}_{n} + \sum_{r>0} r \psi^{\mu}_{-r} \psi^{\mu}_{r} \right) + \\
+ 2\pi \sum_{i=1,2,3} \left(\sum_{n=-\infty}^{\infty} \beta^{i}_{n+m\nu_i} \bar{\beta}^{i}_{-n-m\nu_i} + \widetilde{\beta}^{i}_{n-m\nu_i} \widetilde{\beta}^{i}_{n-m\nu_i} \widetilde{\bar{\beta}}^{i}_{-n+m\nu_i} + \right. \\
\left. + \sum_{r} (r - m\nu_i) \lambda^{i}_{-r+m\nu_i} \bar{\lambda}^{i}_{r+m\nu_i} + (r + m\nu_i) \widetilde{\lambda}^{i}_{-r-m\nu_i} \widetilde{\bar{\lambda}}^{i}_{r+m\nu_i} \right) + 2\pi C_0 \,,
\end{aligned}
\tag{A.10}
$$

where C_0 is the zero point energy which can be easily computed in the different sectors using the fact that for a complex boson transforming as $e^{2\pi i a}$ it is $-\frac{1}{12} + \frac{1}{2} a(1-a)$ and opposite for a complex fermion. Similar results can be obtained for the $\mathbb{C}^3/\mathbb{Z}_N \times \mathbb{Z}_N$ orbifold.

In computing open string partition functions and the matrix elements of the boundary states, it is convenient to introduce the following functions

$$
\begin{aligned}
\eta(\tau) &= q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \,, \\
\vartheta_1(\nu, \tau) &= 2 \exp(\pi i \tau/4) \sin(\pi\nu) \prod_{n=1}^{\infty} (1 - q^n)(1 - e^{2\pi i \nu} q^n)(1 - e^{-2\pi i \nu} q^n) \,, \\
\vartheta_2(\nu, \tau) &= 2 \exp(\pi i \tau/4) \cos(\pi\nu) \prod_{n=1}^{\infty} (1 - q^n)(1 + e^{2\pi i \nu} q^n)(1 + e^{-2\pi i \nu} q^n) \,, \\
\vartheta_3(\nu, \tau) &= \prod_{n=1}^{\infty} (1 - q^n)(1 + e^{2\pi i \nu} q^{n-1/2})(1 + e^{-2\pi i \nu} q^{n-1/2}) \,, \\
\vartheta_4(\nu, \tau) &= \prod_{n=1}^{\infty} (1 - q^n)(1 - e^{2\pi i \nu} q^{n-1/2})(1 - e^{-2\pi i \nu} q^{n-1/2}) \,.
\end{aligned}
\tag{A.11}
$$

In order to compare the answer in the open string channel with the closed string one, we need the modular properties of ϑ functions

$$
\begin{aligned}
&\eta(\tau) = (-i\tau)^{-1/2} \eta\left(-\frac{1}{\tau}\right), \qquad \vartheta_1(\tau) = i(-i\tau)^{-1/2} e^{-\pi i \nu^2} \vartheta_1\left(\frac{\nu}{\tau}, -\frac{1}{\tau}\right), \\
&\vartheta_2(\tau) = (-i\tau)^{-1/2} e^{-\pi i \nu^2} \vartheta_4\left(\frac{\nu}{\tau}, -\frac{1}{\tau}\right), \qquad \vartheta_3(\tau) = (-i\tau)^{-1/2} e^{-\pi i \nu^2} \vartheta_3\left(\frac{\nu}{\tau}, -\frac{1}{\tau}\right), \\
&\vartheta_4(\tau) = (-i\tau)^{-1/2} e^{-\pi i \nu^2} \vartheta_2\left(\frac{\nu}{\tau}, -\frac{1}{\tau}\right).
\end{aligned}
\tag{A.12}
$$

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